LECTURE 11
CHAPTER 3
INTEGER FUNCTIONS

PART 1: Floors and Ceilings

PART 2: Floors and Ceilings Applications
PART 1
Floors and Ceilings
Floor and Ceiling Definitions

Floor Definition
For any $x \in \mathbb{R}$ we define

$$\lfloor x \rfloor = \text{the greatest integer less than or equal to } x$$

Ceiling Definition
For any $x \in \mathbb{R}$ we define

$$\lceil x \rceil = \text{the least (smallest) integer greater than or equal to } x$$
Floor and Ceiling Definitions

**Definitions** written Symbolically

**Floor**

\[
\lfloor x \rfloor = \max\{ a \in \mathbb{Z} : a \leq x \}
\]

**Ceiling**

\[
\lceil x \rceil = \min\{ a \in \mathbb{Z} : a \geq x \}
\]
Floor and Ceiling Basics

Remark: we use, after the book the notion of max, min elements instead of the least (smallest) and greatest elements because for the Posets $P_1$, $P_2$ we have that

\[ P_1 = (\{ a \in \mathbb{Z} : a \leq x \}, \leq) \] has unique max element that is the greatest and

\[ P_2 = (\{ a \in \mathbb{Z} : a \geq x \}, \geq) \] has unique min element that is the least (smallest)
Floor and Ceiling Basics

Fact 1
For any $x \in \mathbb{R}$, $\lfloor x \rfloor$ and $\lceil x \rceil$ exist and are unique.

We define functions

Floor

$$f_1 : \mathbb{R} \rightarrow \mathbb{Z}$$

$$f_1(x) = \lfloor x \rfloor = \max\{a \in \mathbb{Z} : a \leq x\}$$

Ceiling

$$f_2 : \mathbb{R} \rightarrow \mathbb{Z}$$

$$f_2(x) = \lceil x \rceil = \min\{a \in \mathbb{Z} : a \geq x\}$$
Floor and Ceiling Basics

Graphs of $f_1, f_2$
Properties of $\lfloor x \rfloor$ and $\lceil x \rceil$

1. $\lfloor x \rfloor = x$ if and only if $x \in \mathbb{Z}$

2. $\lceil x \rceil = x$ if and only if $x \in \mathbb{Z}$

3. $x - 1 < \lfloor x \rfloor \leq \lceil x \rceil < x + 1$ $x \in \mathbb{R}$

4. $\lfloor -x \rfloor = -\lceil x \rceil$ $x \in \mathbb{R}$
Properties of $\lfloor x \rfloor$ and $\lceil x \rceil$

5. $\lceil -x \rceil = -\lfloor x \rfloor \quad x \in R$

6. $\lceil x \rceil - \lfloor x \rfloor = [x \not\in Z]$ characteristic function of $x \not\in Z$

we re-write 6. as follows

7. $\lfloor x \rfloor - \lfloor x \rfloor = 0$ for $x \in Z$

$\lfloor x \rfloor - \lfloor x \rfloor = 1$ for $x \not\in Z$
Properties of $\lfloor x \rfloor$ and $\lceil x \rceil$

8. $\lfloor x \rfloor = n$ if and only if $n \leq x < n + 1$ for $x \in R$, $n \in Z$

9. $\lceil x \rceil = n$ if and only if $x - 1 < n \leq x$ for $x \in R$, $n \in Z$
Properties of $\lfloor x \rfloor$ and $\lceil x \rceil$

10. $\lfloor x \rfloor = n$ if and only if $n - 1 < x \leq n$ for $x \in R$, $n \in Z$

11. $\lceil x \rceil = n$ if and only if $x \leq n < x + 1$ for $x \in R$, $n \in Z$

12. $\lfloor x + n \rfloor = \lfloor x \rfloor + n$ and $\lceil x + n \rceil = \lceil x \rceil + n$ for $x \in R$, $n \in Z$
Some Proofs

Proof of

12. \(\lfloor x + n \rfloor = \lfloor x \rfloor + n\) for \(x \in \mathbb{R}, \ n \in \mathbb{Z}\)

Directly from definition we have that

\[
\lfloor x \rfloor \leq x < \lfloor x \rfloor + 1
\]

Adding \(n\) to all sides we get

\[
\lfloor x \rfloor + n \leq x + n < \lfloor x \rfloor + n + 1
\]

Applying

8. \(\lfloor x \rfloor = m\) if and only if \(m \leq x < m + 1\) for \(x \in \mathbb{R}, \ m \in \mathbb{Z}\)

for \(m = \lfloor x \rfloor + n\) we get \(\lfloor x + n \rfloor = m\), i.e.

\[
\lfloor x + n \rfloor = \lfloor x \rfloor + n
\]
Some Proofs

**Observe** that it is **not true** that for all $x \in \mathbb{R}$, $n \in \mathbb{Z}$

$$\lfloor nx \rfloor = n \lfloor x \rfloor$$

Take $n = 2$, $x = \frac{1}{2}$ and we get that

$$\left\lfloor 2 \cdot \frac{1}{2} \right\rfloor = 1 \neq 2 \left\lfloor \frac{1}{2} \right\rfloor = 0$$
More Properties of $[x]$ and $\lceil x \rceil$

In all properties $x \in \mathbb{R}$, $n \in \mathbb{Z}$

13. $x < n$ if and only if $[x] < n$

14. $n < x$ if and only if $n < \lceil x \rceil$

15. $x \leq n$ if and only if $[x] \leq n$

16. $n \leq x$ if and only if $n \leq [x]$
Some Proofs

Proof of 13. $x < n$ if and only if $\lfloor x \rfloor < n$

Let $x < n$

We know that $\lfloor x \rfloor \leq x$ so $\lfloor x \rfloor \leq x < n$

and hence $\lfloor x \rfloor < n$

Let $\lfloor x \rfloor < n$

By property 3. $x - 1 < \lfloor x \rfloor \leq \lfloor x \rfloor < x + 1$, $x \in R$

$x - 1 < \lfloor x \rfloor$, i.e $x < \lfloor x \rfloor + 1$

But $\lfloor x \rfloor < n$, so $\lfloor x \rfloor + 1 \leq n$ and

$$x < \lfloor x \rfloor + 1 \leq n$$

Hence $x < n$ what ends the proof
Fractional Part of \( x \)

**Definition**
We define:

\[
\{ x \} = x - \lfloor x \rfloor
\]

\( \{ x \} \) is called a **fractional** part of \( x \)

\( \lfloor x \rfloor \) is called the **integer** part of \( x \)

By definition

\[
0 \leq \{ x \} < 1
\]

and we write

\[
x = \lfloor x \rfloor + \{ x \}
\]
Fractional Part of $x$

**Fact 2**

IF $x = n + \Theta$, $n \in \mathbb{Z}$ and $0 \leq \Theta < 1$

THEN $n = \lfloor x \rfloor$ and $\Theta = \{x\}$

**Proof**

Let $x = n + \Theta$, $\Theta \in [0, 1)$. We get by 12.

$\lfloor x \rfloor = \lfloor n + \Theta \rfloor = n + \lfloor \Theta \rfloor = n$ and

$$x = n + \Theta = \lfloor x \rfloor + \Theta = \lfloor x \rfloor + \{x\}$$

so $\Theta = \{x\}$
Properties

We have proved in 12.

\[ [x + n] = [x] + n \text{ for } x \in R, \ n \in Z \]

**Question:** What happens when we consider 

\[ [x + y] \text{ where } x \in R \text{ and } y \in R \]

Is it possible (and when it is possible) that for any \( x, y \in R \)

\[ [x + y] = [x] + [y] \]
Properties

Consider

\[ x = [x] + \{x\}, \quad \text{and} \quad y = [y] + \{y\} \]

We evaluate using 12. \([x + n] = [x] + n\)

\([x + y] = [x] + [y] + \{x\} + \{y\} = [x] + [y] + [\{x\} + \{y\}]\]

By definition 0 \leq \{x\} < 1 and 0 \leq \{y\} < 1 so we have that

\[ 0 \leq \{x\} + \{y\} < 2 \]

Hence we have proved the following property
Properties

Fact 3
For any \( x, y \in \mathbb{R} \)

\[
[x + y] = [x] + [y] \quad \text{when} \quad 0 \leq \{x\} + \{y\} < 1
\]

and

\[
[x + y] = [x] + [y] + 1 \quad \text{when} \quad 1 \leq \{x\} + \{y\} < 2
\]
Examples

Example 1
Find $\lceil \log_2 35 \rceil$

Observe that $2^5 < 35 \leq 2^6$

Taking log with respect to base 2, we get

$$5 < \log_2 35 \leq 6$$

We use property

10. $\lfloor x \rfloor = n$ if and only if $n - 1 < x \leq n$

and get

$$\lceil \log_2 35 \rceil = 6$$
Examples

Example 2
Find $\lceil \log_2 32 \rceil$

Observe that $2^4 < 32 \leq 2^5$

Taking log with respect to base 2, we get

$$4 < \log_2 32 \leq 5$$

We use property 10. and get

$$\lceil \log_2 32 \rceil = 5$$
Examples

Example 3
Find \( \lfloor \log_2 35 \rfloor \)

Observe that \( 2^5 \leq 35 < 2^6 \)

Taking \( \log \) with respect to base 2 , we get

\[
5 \leq \log_2 32 < 6
\]

We use property

8. \( \lfloor x \rfloor = n \) if and only if \( n \leq x < n + 1 \)

and we get

\[
\lfloor \log_2 32 \rfloor = 5 = \lceil \log_2 32 \rceil
\]
Observation

Observe that 35 has 6 digits in its binary representation 
$35 = (1000011)_2$ and $\lceil \log_2 35 \rceil = 6$

Question

Is the number of digits in binary representation of $n$ always equal $\lceil \log_2 n \rceil$?

Answer: NO, it is not true

Consider $32 = (1000000)_2$

32 has 6 digits in its binary representation but 

$\lceil \log_2 32 \rceil = 5 \neq 6$
Small Problem

**Question:** Can we develop a connection (formula) between $\lfloor \log_2 n \rfloor$ and number of digits (m) in the binary representation of $n$ ($n > 0$)?

**Answer:** YES
Small Problem Solution

Let \( n \neq 0, n \in \mathbb{N} \) be such that it has \( m \) bits in binary representation. Hence, by definition we have

\[
n = a_{m-1}2^{m-1} + \ldots + a_0
\]

and

\[
2^{m-1} \leq n < 2^m
\]

So we get solution

\[
m - 1 \leq \log_2 n < m \quad \text{if and only if} \quad \lfloor \log_2 n \rfloor = m - 1
\]
Small Fact and Exercise

We have proved the following

Fact 4
For any \( n \neq 0, \ n \in N \) such such that it has \( m \) bits in binary representation we have that

\[
\lfloor \log_2 n \rfloor = m - 1
\]

Example
Take \( n = 35, \ m = 6 \) so \( \lfloor \log_2 35 \rfloor = 6 - 1 = 5 \)
Take \( n = 32, \ m = 6 \) so we get \( \lfloor \log_2 32 \rfloor = 6 - 1 = 5 \)

Exercise  Develop similar formula for \( \lceil \log_2 n \rceil \)
Another Small Fact

Fact 5
For any \( x \in \mathbb{R}, \ x \geq 0 \) the following property holds

\[
\left\lfloor \sqrt{\left\lfloor x \right\rfloor} \right\rfloor = \left\lfloor \sqrt{x} \right\rfloor
\]

Proof
Take \( \left\lfloor \sqrt{\left\lfloor x \right\rfloor} \right\rfloor \)

We proceed as follows

**First** we get rid of the **outside** \( \left\lfloor \right\rfloor \) and **then** of the square root and of the **inside** \( \left\lfloor \right\rfloor \)
Proof

Let \( m = \left\lfloor \sqrt{\lfloor x \rfloor} \right\rfloor \)

By property

8.  \( \lfloor x \rfloor = n \) if and only if \( n \leq x < n+1 \)

we get that

\[
  m = \left\lfloor \sqrt{\lfloor x \rfloor} \right\rfloor \quad \text{if and only if} \quad m \leq \sqrt{\lfloor x \rfloor} < m+1
\]

Squaring all sides of the inequality we get

(\(\star\))  \( m = \left\lfloor \sqrt{\lfloor x \rfloor} \right\rfloor \) if and only if  \( m^2 \leq \lfloor x \rfloor < (m+1)^2 \)
Proof

We proved that

\[ m = \lfloor \sqrt{\lfloor x \rfloor} \rfloor \quad \text{if and only if} \quad m^2 \leq \lfloor x \rfloor < (m+1)^2 \]

Using property

16. \quad n \leq x \quad \text{if and only if} \quad n \leq \lfloor x \rfloor

on the left of inequality in \((\star)\) and property

13. \quad x < n \quad \text{if and only if} \quad \lfloor x \rfloor < n

on the right side of inequality in \((\star)\) we get

\[ m = \lfloor \sqrt{\lfloor x \rfloor} \rfloor \quad \text{if and only if} \quad m^2 \leq x < (m+1)^2 \]
Proof

We already proved that

\[(\star\star) \quad m = \left\lfloor \sqrt{\lfloor x \rfloor} \right\rfloor \quad \text{if and only if} \quad m^2 \leq x < (m+1)^2\]

Now we retrace our steps backwards. First taking \(\sqrt{x}\) on all sides of inequality \((\star\star)\) (all components are \(\geq 0\)), we get

\[m = \left\lfloor \sqrt{\lfloor x \rfloor} \right\rfloor \quad \text{if and only if} \quad m \leq \sqrt{x} < m+1\]

We use now the property

8. \(\lfloor x \rfloor = n \quad \text{if and only if} \quad n \leq x < n+1\)

and get

\[m = \left\lfloor \sqrt{\lfloor x \rfloor} \right\rfloor \quad \text{if and only if} \quad \lfloor \sqrt{x} \rfloor = m\]

and hence

\[\left\lfloor \sqrt{\lfloor x \rfloor} \right\rfloor = \lfloor \sqrt{x} \rfloor\]

It ends the proof
Exercise

Write a proof of

\[
\left\lfloor \sqrt{\left\lceil x \right\rceil} \right\rfloor = \left\lfloor \sqrt{x} \right\rfloor
\]

Question

How can we \textbf{GENERALIZE} our just proven properties for other functions then \( f(x) = \sqrt{x} \) ?

For which functions \( f = f(x) \) (class of which functions?) the following holds

\[
\left\lfloor f(\left\lfloor x \right\rfloor) \right\rfloor = \left\lfloor f(x) \right\rfloor
\]

and

\[
\left\lceil f(\left\lfloor x \right\rfloor) \right\rceil = \left\lceil f(x) \right\rceil
\]
Generalization

Here is a proper generalization of the **Fact 4**

**Fact 5**

Let \( f : R' \rightarrow R \) where \( R' \subseteq R \) is the domain of \( f \)

**IF** \( f = f(x) \) is continuous, monotonically increasing on its domain \( R' \), and additionally has the following property \( P \)

\[
P \quad \text{if} \quad f(x) \in Z \quad \text{then} \quad x \in Z
\]

**THEN** for all \( x \in R' \) for which the property \( P \) holds we have that

\[
\lfloor f(\lfloor x \rfloor) \rfloor = \lfloor f(x) \rfloor
\]

and

\[
\lceil f(\lceil x \rceil) \rceil = \lceil f(x) \rceil
\]
Fact 5 Proof

Proof
We want to show that under assumption that \( f \) is continuous, monotonic, increasing on its domain \( R' \) the property
\[
\lceil f(\lceil x \rceil) \rceil = \lceil f(x) \rceil
\]
holds for all \( x \in R' \) for which the property \( P \) holds.

Case 1  take \( x = \lfloor x \rfloor \)
We get
\[
\lfloor f(x) \rfloor = \lfloor f(\lfloor x \rfloor) \rfloor
\]
is trivial as in this case we have that \( x \in \mathbb{Z} \)
Fact 5 Proof

Case 2: take $x \neq \lfloor x \rfloor$

By definition $x < \lfloor x \rfloor$ and function $f$ is monotonically increasing so we have

$$f(x) < f(\lfloor x \rfloor)$$

By the fact that $\lfloor \rfloor$ is non-decreasing, i.e.

If $x < y$ then $\lfloor x \rfloor \leq \lfloor y \rfloor$

we get

$$\lfloor f(x) \rfloor \leq \lfloor f(\lfloor x \rfloor) \rfloor$$

Now we show that $<$ is impossible

Hence we will have $=$
Fact 5 Proof

Assume

\[ [f(x)] < [f([x])] \]

Since \( f \) is continuous, then there is \( y \) such that

\[ f(y) = [f(x)] \]

and

\[ (\star) \quad x \leq y < [x] \]

But \( f(y) = [f(x)] \), i.e. \( f(y) \in \mathbb{Z} \) hence by property \( P \) we get

\[ (\star\star) \quad y \in \mathbb{Z} \]

Observe that \((\star)\) and \((\star\star)\) are contradictory as there is no \( y \in \mathbb{Z} \) between \( x \) and \([x]\) and this ends the proof
Exercises

Exercise 1
Prove the first part of the Fact 5, i.e.

\[ \left\lfloor \sqrt{\left\lfloor f(x) \right\rfloor} \right\rfloor = \left\lfloor \sqrt{f(x)} \right\rfloor \]

Exercise 2
Prove that for any \( x \in \mathbb{R}, \ n, m \in \mathbb{Z} \)

1. \[ \left\lfloor \frac{x + m}{n} \right\rfloor = \left\lfloor \frac{\lfloor x \rfloor + m}{n} \right\rfloor \]

and

2. \[ \left\lceil \frac{x + m}{n} \right\rceil = \left\lceil \frac{\lfloor x \rfloor + m}{n} \right\rceil \]
Exercise 2 Solution

Let’s prove

1. \[ \left\lfloor \frac{x + m}{n} \right\rfloor = \left\lfloor \frac{\lfloor x \rfloor + m}{n} \right\rfloor \]

Proof for \( \left\lceil \right\rceil \) is carried similarly and is left as an exercise.

Take a function

\[ f(x) = \frac{x + m}{n} \]

for \( n, m \in Z, \ x \in R \)

Observe that

\[ f(x) = \frac{x + m}{n} = \frac{x}{n} + \frac{m}{n} \]

is a line \( f(x) = ax + b \) and hence is continuous, monotonically increasing.
Exercise 2 Solution

We have to check now if the property $P$

$$P \quad \text{if } f(x) \in \mathbb{Z} \quad \text{then} \quad x \in \mathbb{Z}$$

holds for it, i.e. to check if all assumptions of the Fact 5 are fulfilled.

Then by the Fact 5 we will get that

$$\lfloor f(\lfloor x \rfloor) \rfloor = \lfloor f(x) \rfloor$$

i.e.

$$\left\lfloor \frac{\lfloor x \rfloor + m}{n} \right\rfloor = \left\lfloor \frac{x + m}{n} \right\rfloor$$
Exercise 2 Solution

**Proof** that the property $P$ holds for

$$f(x) = \frac{x + m}{n}$$

Assume $f(x) \in \mathbb{Z}$, i.e. there is $k \in \mathbb{Z}$ such that

$$\frac{x + m}{n} = k$$

It means that

$$x + m = nk$$

and

$$x = nk - m \in \mathbb{Z} \quad \text{as} \quad n, k, m \in \mathbb{Z}$$
Integers in the Intervals
Intervals

**Standard Notation** and definition of a Closed Interval

\[ [\alpha, \beta] = \{ x \in R : \alpha \leq x \leq \beta \} \]

**Book Notation**

\[ [\alpha \ldots \beta] = \{ x \in R : \alpha \leq x \leq \beta \} \]

We use book notation, because \([P(x)]\) denotes in the book the characteristic function of \(P(x)\)
Intervals

Closed Interval

\[[\alpha, \beta] = \{x \in \mathbb{R} : \alpha \leq x \leq \beta\} = [\alpha...\beta]\]

Open Interval

\)((\alpha, \beta) = \{x \in \mathbb{R} : \alpha < x < \beta\} = (\alpha...\beta)\]

Half Open Interval

\[[\alpha, \beta) = \{x \in \mathbb{R} : \alpha \leq x < \beta\} = [\alpha...\beta)\]

Half Open Interval

\)((\alpha, \beta] = \{x \in \mathbb{R} : \alpha < x \leq \beta\} = (\alpha...\beta]\)
Integers in the Intervals

Problem
How many integers are there in the intervals?
In other words, for

\[ A = \{ x \in \mathbb{Z} : \alpha \leq x \leq \beta \} \]
\[ A = \{ x \in \mathbb{Z} : \alpha < x \leq \beta \} \]
\[ A = \{ x \in \mathbb{Z} : \alpha \leq x < \beta \} \]
\[ A = \{ x \in \mathbb{Z} : \alpha < x < \beta \} \]

We want to find \(|A|\)
Integers in the Intervals

Solution
We bring our ⌈⌉, ⌊⌋ properties 13. - 16.

13. \( x < n \) if and only if \( \lfloor x \rfloor < n \)

14. \( n < x \) if and only if \( n < \lceil x \rceil \)

15. \( x \leq n \) if and only if \( \lfloor x \rfloor \leq n \)

16. \( n \leq x \) if and only if \( n \leq \lceil x \rceil \)

and we get for \( \alpha, \beta \in R \) and \( n \in Z \)

\[ \alpha \leq n < \beta \] if and only if \( \lfloor \alpha \rfloor \leq n < \lceil \beta \rceil \)

\[ \alpha < n \leq \beta \] if and only if \( \lfloor \alpha \rfloor \leq n < \lceil \beta \rceil \)
Integers in the Intervals

Solution

\[(\alpha...\beta) \text{ contains exactly } \lceil \beta \rceil - \lceil \alpha \rceil \text{ integers}\]

\((\alpha...\beta] \text{ contains exactly } \lceil \beta \rceil - \lfloor \alpha \rfloor \text{ integers}\]

\[[\alpha...\beta] \text{ contains exactly } \lceil \beta \rceil - \lfloor \alpha \rfloor + 1 \text{ integers}\]

We must assume \(\alpha \neq \beta\) to evaluate

\((\alpha...\beta) \text{ contains exactly } \lfloor \beta \rfloor - \lfloor \alpha \rfloor - 1 \text{ integers}\]

We

because \((\alpha...\alpha) = \emptyset\) and can’t contain \(-1\) integers
### Integers in the Intervals

<table>
<thead>
<tr>
<th>INTERVAL</th>
<th>Number of INTEGERS</th>
<th>RESTRICTIONS</th>
</tr>
</thead>
<tbody>
<tr>
<td>$[\alpha ... \beta]$</td>
<td>$\lfloor \beta \rfloor - \lceil \alpha \rceil + 1$</td>
<td>$\alpha \leq \beta$</td>
</tr>
<tr>
<td>$[\alpha ... \beta)$</td>
<td>$\lfloor \beta \rfloor - \lceil \alpha \rceil$</td>
<td>$\alpha \leq \beta$</td>
</tr>
<tr>
<td>$(\alpha ... \beta]$</td>
<td>$\lfloor \beta \rfloor - \lceil \alpha \rceil$</td>
<td>$\alpha \leq \beta$</td>
</tr>
<tr>
<td>$(\alpha ... \beta)$</td>
<td>$\lfloor \beta \rfloor - \lceil \alpha \rceil - 1$</td>
<td>$\alpha &lt; \beta$</td>
</tr>
</tbody>
</table>
Casino Problem
Casino Problem

There is a roulette wheel with 1,000 slots numbered 1 \ldots 1,000.

IF the number $n$ that comes up on a spin is divisible by $\left\lfloor \sqrt[3]{n} \right\rfloor$ what we write as

$$\left\lfloor \sqrt[3]{n} \right\rfloor \mid n$$

THEN $n$ is the **winner**

Reminder

We define divisibility in a standard way:

$k \mid n$ if and only if there exists $m \in \mathbb{Z}$ such that $n = km$
Average Winnings

In the game Casino pays $5 if you are the winner; but the loser has to pay $1.

Can we expect to make money if we play this game?

Let’s compute average winnings, i.e. the amount we win (or lose) per play.

Denote

$W$ - number of winners
$L$ - number of losers and $L = 1000 - W$

Strong Rule: each number comes once during 1000 plays.
Casino Winnings

Under the **Strong Rule** we win $5W$ and lose $L$ dollars and the average winnings in 1000 plays is

$$\frac{5W - L}{1000} = \frac{5W - (1000 - W)}{1000} = \frac{6W - 1000}{1000}$$

We have *advantage* if

$$6W > 1000$$

i.e. when

$$W > 167$$
Casino Winnings

Answer

IF there is 167 or more winners and we play under the

**Strong Rule**: each number *comes once* during 1000 plays

THEN we have the **advantage**, otherwise **Casino wins**
Number of Winners

Problem

How to count the number of winners among 1 to 1000

Method

Use summation

\[ W = \sum_{n=1}^{1000} [n \text{ is a winner}] \]
Reminder of Casino Problem

There is a roulette wheel with 1,000 slots numbered 1 \ldots 1,000

If the number \( n \) that comes up on a spin is divisible by \( \lfloor 3\sqrt{n} \rfloor \), i.e. \( \lfloor 3\sqrt{n} \rfloor | n \)

Then \( n \) is the winner

The summations becomes

\[
W = \sum_{n=1}^{1000} [n \text{ is a winner}] = \sum_{n=1}^{1000} [\lfloor 3\sqrt{n} \rfloor | n]
\]

where we define divisibility in a standard way

\( k | n \) if and only if there exists \( m \in \mathbb{Z} \) such that \( n = km \)
Here are 7 steps of our BOOK solution

1. \[ W = \sum_{n=1}^{1000} \left[ n \text{ is a winner } \right] = \sum_{n=1}^{1000} \left\lfloor \frac{3}{\sqrt{n}} \right\rfloor \cdot n \]

2. \[ W = \sum_{k,n} \left[ k = \left\lfloor \frac{3}{\sqrt{n}} \right\rfloor \right] [k|n] [1 \leq n \leq 1000] \]

3. \[ W = \sum_{k,n,m} \left[ k^3 \leq n < (k+1)^3 \right] [n = km] [1 \leq n \leq 1000] \]

4. \[ W = 1 + \sum_{k,m} \left[ k^3 \leq km < (k+1)^3 \right] [1 \leq k < 10] \]

5. \[ W = 1 + \sum_{k,m} \left[ m \in \left[ k^2 \ldots \frac{(k+1)^3}{k} \right) \right] [1 \leq k < 10] \]

6. \[ W = 1 + \sum_{1 \leq k < 10} \left( \left\lfloor k^2 + 3k + 3 + \frac{1}{k} \right\rfloor - \left\lfloor k^2 \right\rfloor \right) \]

7. \[ W = 1 + \sum_{1 \leq k < 10} (3k + 4) = 1 + \frac{7 + 31}{2} \cdot 9 = 172 \]
Class Problem

Here are the BOOK comments

1. This derivation merits careful study

2. The only "difficult" maneuver is the decision between lines 3 and 4 to treat $n = 1000$ as a special case

3. The inequality $k^3 \leq n < (k + 1)^3$ does not combine easily with $1 \leq n \leq 1000$ when $k = 10$
Book Solution Comments

Class Problem

Write down explanation of each step with detailed justifications (Facts, definitions) why they are correct

By doing so fill all gaps in the proof that

\[ W = \sum_{n=1}^{1000} \left\lfloor \frac{3}{\sqrt[3]{n}} \right\rfloor \cdot n = 172 \]

This problem can also appear on your tests
QUESTIONS about Book Solution

Here are questions to answer about the steps in the BOOK solution

1. \[ W = \sum_{n=1}^{1000} [n \text{ is a winner}] = \sum_{n=1}^{1000} \left\lfloor \frac{3}{n} \right\rfloor | n \]

Q1 Explain why \([n \text{ is a winner}] = \left\lfloor \frac{3}{n} \right\rfloor | n\]

2. \[ W = \sum_{k,n} \left[ k = \left\lfloor \frac{3}{n} \right\rfloor \right] [k|n] [1 \leq n \leq 1000] \]

Q2 Explain why and how we have changed a sum \(\sum_{n=1}^{1000}\) into a sum \(\sum_{k,n}\) and

\[ \sum_{n=1}^{1000} \left\lfloor \frac{3}{n} \right\rfloor | n \] = \(\sum_{k,n} \left[ k = \left\lfloor \frac{3}{n} \right\rfloor \right] [k|n] [1 \leq n \leq 1000] \]
QUESTIONS about Book Solution

3  \[ W = \sum_{k,n,m} \left[ k^3 \leq n < (k + 1)^3 \right] [n = km][1 \leq n \leq 1000] \]

Q3  Explain why

\[ k = \lfloor \sqrt[3]{n} \rfloor [k \mid n] = \left[ k^3 \leq n < (k + 1)^3 \right] [n = km] \]

Explain why and how we have changed sum \( \sum_{k,n} \) into a sum \( \sum_{k,n,m} \)
QUESTIONS about Book Solution

4 \[ W = 1 + \sum_{k,m} \left[ k^3 \leq km < (k+1)^3 \right] \quad [1 \leq k < 10] \]

Q4 There are three sub-questions; the last one is one of the book questions

1. Explain why
\[
\left[ k^3 \leq n < (k+1)^3 \right] [n = km] [1 \leq n \leq 1000] = \\
\left[ k^3 \leq km < (k+1)^3 \right] [1 \leq k < 10]
\]

2. Explain why and how we have changed sum \( \sum_{k,n,m} \) into a sum \( \sum_{k,m} \)

3. Explain HOW and why we have got \( 1+ \sum_{k,m} \)
QUESTIONS about Book Solution

5. \( W = 1 + \sum_{k,m} \left[ m \in \left[ k^2 \ldots \frac{(k+1)^3}{k} \right] \right] [1 \leq k < 10] \)

Q5 Explain transition

\[
\left[ k^3 \leq km < (k + 1)^3 \right] = \left[ m \in \left[ k^2 \ldots \frac{(k+1)^3}{k} \right] \right]
\]
QUESTIONS about Book Solution

6 \quad W = 1 + \sum_{1 \leq k < 10} \left( \left\lceil k^2 + 3k + 3 + \frac{1}{k} \right\rceil - \left\lceil k^2 \right\rceil \right)

Q6 \quad Explain (prove) why

\sum_{k,m} \left[ m \in \left[ k^2 \ldots \frac{(k+1)^3}{k} \right] \right] [1 \leq k < 10] =

\sum_{1 \leq k < 10} \left( \left\lceil k^2 + 3k + 3 + \frac{1}{k} \right\rceil - \left\lceil k^2 \right\rceil \right)

Observe that \left[ m \in \left[ k^2 \ldots \frac{(k+1)^3}{k} \right] \right] is a characteristic function and \left( \left\lceil k^2 + 3k + 3 + \frac{1}{k} \right\rceil - \left\lceil k^2 \right\rceil \right) is an integer
QUESTIONS about Book Solution

7 \[ W = 1 + \sum_{1 \leq k < 10} (3k + 4) = 1 + \frac{7 + 31}{2} \cdot 9 = 172 \]

Q7 Explain (prove) why

\[ (\lceil k^2 + 3k + 3 + \frac{1}{k} \rceil - \lfloor k^2 \rfloor) = (3k + 4) \]

Before we giving answers to Q1 - Q7 we need to review some of the SUMS material