cse547/ams547 MIDTERM 2 Spring 2010

100pts + 10 extra points

NAME **ID:** **ams/cs**

Each problem 1-5 is worth 20pts.

Useful Formulas sheet is attached!!!

QUESTION 1

Part 1 Prove that the sequence $a_n = n!$ grows faster then the sequence $b_n = c^n$ for any $c > 0$.

Hint: use d'Alambert Criterium for a proper infinite sum and a proper theorem about infinite sums.

Part 2 We know that the Harmonic series $\Sigma_{n=1}^{\infty} \frac{1}{n}$ **diverges**. Use this information and **Cauchy Criterium** to prove that

$$\lim_{n \to \infty} \sqrt[n]{n} = 1.$$

QUESTION 2 Give a direct proof from proper properties (list which) of the following fact.

For all $x \in R, x > 0$

$$\lfloor \sqrt{\lfloor x \rfloor} \rfloor = \lfloor \sqrt{x} \rfloor$$

QUESTION 3 Prove the following

Spectrum Partition Theorem Let $\alpha, \beta > 0, \alpha, \beta \in R - Q$ be such that

$$\frac{1}{\alpha} + \frac{1}{\beta} = 1.$$

THEN the sets $A = spec(\alpha)$ and $B = spec(\beta)$ form a partition of $Z^+ = N - \{0\}$, i.e.

1. $A \neq \emptyset, \quad B \neq \emptyset$
2. $A \cap B = \emptyset$
3. $A \cup B = Z^+$.

QUESTION 4 Show that the nth element of the sequence:

$$1, 2, 2, 3, 3, 3, 4, 4, 4, 4, 5, 5, 5, 5, 5, ...$$

is $\lfloor \sqrt{2n} + \frac{1}{2} \rfloor$.

Hint: Let P(x) represent the position of the last occurrence of x in the sequence.
Use the fact that $P(x) = \frac{x(x+1)}{2}$.

Let the nth element be m. You need to find m.

QUESTION 5 Prove the following theorems.

Theorem 1 Let $m, n, k \in Z^{+} - \{0\}$.

IF $k|mn$ and $k \perp m$ (it means k, m are relatively prime), THEN $k|n$.

Theorem 2 When a number is relatively prime to each of several numbers, it is relatively prime to their product.

QUESTION 6 (EXTRA CREDIT- 10pts)

Show that

$\lfloor (n+1)^2 n! e \rfloor mod n = 2 mod n$, for all $n \in N$.

Hint: use the following

$$e = \sum_{k \geq 0} \frac{1}{k!},$$

and represent $(n+1)^2 n! e$ as

$$(n+1)^2 n! e = A_n + (n+1)^2 + (n+1) + B_n$$

for certain A_n, B_n such that

$$\forall n \in N (A_n \in Z), \quad \forall n \in N \exists k \in Z (A_n = nk), \quad \forall n \in N (0 \leq B_n < 1)$$

EXTRA SPACE

EXTRASPACE

1 Properties

$$\lfloor x \rfloor = x \iff x \in Z, \qquad \lceil x \rceil = x \iff x \in Z$$

$$x - 1 < \lfloor x \rfloor \leq x \leq \lceil x \rceil < x + 1$$

$$\lfloor -x \rfloor = -\lceil x \rceil, \qquad \lceil -x \rceil = -\lfloor x \rfloor$$

$$\lceil x \rceil - \lfloor x \rfloor = 0 \ if \ x \in Z, \qquad \lceil x \rceil - \lfloor x \rfloor = 1 \ if \ x \notin Z$$

$$\lfloor x \rfloor = n \iff n \leq x < n + 1$$

$$\lfloor x \rfloor = n \iff x - 1 < n \leq x$$

$$\lceil x \rceil = n \iff n - 1 < x \leq n$$

$$\lceil x \rceil = n \iff x \leq n < x + 1$$

$$\lfloor x + n \rfloor = \lfloor x \rfloor + n$$

$$x < n \iff \lfloor x \rfloor < n$$

$$n < x \iff n < \lceil x \rceil$$

$$x \leq n \iff \lceil x \rceil \leq n$$

$$n \leq x \iff n \leq \lfloor x \rfloor$$

$$[\alpha \ldots \beta] \ contains \ \lfloor \beta \rfloor - \lceil \alpha \rceil + 1 \ integers, \ for \ \alpha \leq \beta$$

$$[\alpha \ldots \beta) \ contains \ \lceil \beta \rceil - \lceil \alpha \rceil \ integers, \ for \ \alpha \leq \beta$$

$$(\alpha \ldots \beta] \ contains \ \lfloor \beta \rfloor - \lfloor \alpha \rfloor \ integers, \ for \ \alpha \leq \beta$$

$$(\alpha \ldots \beta) \ contains \ \lceil \beta \rceil - \lfloor \alpha \rfloor - 1 \ intergers, \ for \ \alpha < \beta$$

$$x = y \left\lfloor \frac{x}{y} \right\rfloor + x \bmod y$$