Let's look at an example of a sum

$$\sum_{Q(k,j)} a_{k,j}$$

for

$$a_{k,j} = \frac{1}{k-j}$$

and

$$Q(k,j) = (1 \leq k \leq n) \cap (1 \leq j \leq n) \cap (j < k)$$

$$= P(k,j) \cap (j < k)$$

We want to evaluate (close $f$)

$$\sum_{P(k,j) \cap (j < k)} \frac{1}{k-j} = \sum_{1 \leq k \leq n, 1 \leq j \leq n, j < k} \frac{1}{k-j} = S_m$$

$s_1 = 0, s_2 = 1, s_3 = \frac{1}{2-1} + \frac{1}{3-1} + \frac{1}{3-2} = \frac{5}{2}$
\[ S_m = \sum_{1 \leq k \leq n} \frac{1}{k-\delta} \]

\[ P(k, \delta) = \left( \sum_{1 \leq \delta \leq n} \right) \cap \left( 1 \leq j \leq k \right) \]

0. Simplify the sum boundaries

PROVE

\[ P(k, j) \cap j \leq k \equiv \left( 1 \leq k \leq n \right) \cap \left( 1 \leq j \leq k \right) \]

Evaluate

\[ (1 \leq k \leq n) \cap (1 \leq j \leq n) \cap (j \leq k) \]

\[ = (1 \leq k \leq n) \cap (1 \leq j \leq k) \cap (j \leq k) \cap (j \leq n) \]

\[ \downarrow \]

\[ (1 \leq k \leq n) \cap (1 \leq j \leq k) \]

We proved

\[ P(k, j) \cap (j < k) \Rightarrow (1 \leq k \leq n) \cap (1 \leq j < k) \]
We have to show now

\[(1 \leq k \leq n) \cap (1 \leq j \leq k) \Rightarrow P(k, j) \cap (j \leq k)\]

\[
(1 \leq k \leq n) \cap (1 \leq j \leq n) \cap (j \leq k) \Rightarrow P(k, j) \cap (1 \leq j)
\]

\[
(1 \leq k \leq n) \cap (1 \leq j \leq n) \cap (j \leq k) \Rightarrow P(k, j) \cap (1 \leq j)
\]

\[
= (1 \leq k \leq n) \cap (1 \leq j \leq n) \cap (j \leq k)
\]

\[
\text{by similar method we get}
\]

1. \[P(k, j) \cap (j \leq k) = (1 \leq j \leq n) \cap (j \leq k)\]

2. \[P(k, j) \cap (j \leq k) = (1 \leq j \leq n) \cap (j \leq k)\]
STEP 1: USE 1 TO EVALUATE $S_m$

$$S_m = \sum_{p(kj)} \frac{1}{k-j} \delta_{k-j} = \sum_{k=1}^{m} \left( \sum_{1 \leq j < k} \frac{1}{k-j} \right)$$

$$(k-j) = \delta$$

Simplify the boundary

$$1 \leq k-j < k = (1 \leq k-j) \cap (k-j < k)$$

$$= (1 + j \leq k) \cap (-j < 0) \cap j \neq 0$$

$$= (j \leq k-1) \cap (j \geq 0) \cap j \neq 0$$

$$= \left(1 \leq j \leq k-1\right)$$

$$S_m = \sum_{k=1}^{m} \left( \sum_{j=1}^{k-1} \frac{1}{\delta} \right) = \sum_{k=1}^{m} H_{k-1}$$
STEP 2: USE 2 TO EVALUATE $S_n$

$S_n = \sum_{1 \leq j \leq n} \left( \sum_{j \leq k \leq n} \frac{1}{k-j} \right) \quad \text{Substitute } k-j := K$

$= \sum_{j=1}^{n} \left( \sum_{j < k-j \leq n} \frac{1}{k} \right) \quad \text{Harmonic } H \, \frac{n-j}{n-j}$

$= \sum_{j=1}^{m} \left( \sum_{k=1}^{m-j} \frac{1}{k} \right) \quad \text{Substitute } m-j := j$

$= \sum_{1 \leq j \leq n} \left( \sum_{1 \leq n-j \leq n} H_{n-j} \right) \quad 1 \leq n-j \leq n$

$= \sum_{j=0}^{m-1} H_{n-j} \quad (1 \leq n-j \leq n) \land (n-j \leq n)$

$\quad \land (j \leq n-1) \land (j \geq 0)$

Still hard to evaluate: NOT CLOSED FORM
Step 3

\[ S_n = \sum_{1 \leq j < k \leq n} \frac{1}{k-j} \]

\[ = \sum_{1 \leq j < k+j \leq n} \frac{1}{k} \]

\[ = \sum_{1 \leq j < k+j \leq n} \frac{1}{k} \]

**Prove**

1. \(1 \leq j < k+j \leq n = (1 \leq k \leq n-1) \cap (1 \leq j \leq n-k)\)

**Evaluate**

\[ S_n = \sum_{1 \leq j < k+j \leq n} \frac{1}{k} = \sum_{1 \leq k \leq n-1} \sum_{1 \leq j \leq n-k} \frac{1}{k} \]
EVALUATE:

\[
(1 \leq j < k+j \leq n) = (1 \leq j) \land (1 \leq n) \\
\land (j \leq n-k) \land (j < k+j \leq n) \\
(k+j \leq n) \\
= (1 \leq j \leq n-k) \land (0 < k \leq n-j)
\]

Look at \(0 < k \leq n-j\) for \(j = 1, 2, \ldots, n-k\)

We get \(1 \leq k \leq n-1\)

Hence

\[
(1 \leq j < k+j \leq n) = (1 \leq j \leq n-k) \\
\land (1 \leq k \leq n-1)
\]
\[ S_n = \sum_{1 \leq k \leq n-1} \frac{1}{k} = \sum_{k=1}^{n-1} \left( \sum_{1 \leq j \leq n-k} \frac{1}{k} \right) \]

\[ = \sum_{k=1}^{n-1} \frac{1}{k} \sum_{j=1}^{n-k} 1 \]

\[ = \frac{1}{k} (n-k) = \frac{m}{k} - 1 \]

\[ S_m = \sum_{k=1}^{m} \frac{1}{k} (n-k) \]

\[ = m H_{n-1} - n + 1 \]

\[ = m \left( H_n - \frac{1}{n} \right) - n + 1 \]

\[ = m H_n - 1 - n + 1 \]

\[ (\text{Our}) \quad \text{Closed formula} \]

\[ S_n = m H_m - n \]
\[ S_m = \sum_{k=1}^{m} \left( \sum_{1 \leq j \leq m-k} \frac{1}{k} \right) \]

= \sum_{k=1}^{m} \left( \sum_{j=1}^{m-k} \frac{1}{k} \right)

= \sum_{k=1}^{m} \frac{m-k}{k} \left( \sum_{j=1}^{k} \frac{1}{j} \right)

= \sum_{k=1}^{m} \frac{1}{k} \left( \sum_{j=1}^{m-k} \frac{1}{j} \right)

= \sum_{k=1}^{m} \frac{1}{k} \left( \sum_{j=1}^{k} \frac{1}{j} \right)

= \sum_{k=1}^{m} \left( \frac{m}{k} - 1 \right)

= m \sum_{k=1}^{m} \frac{1}{k} - \sum_{k=1}^{m} 1

= n \cdot H_m - n

\[ S_m = n \cdot H_m - n \]
We proved also (before)

\[ S_m = \sum_{k=1}^{m} H_k \]

we get

\[ \sum_{k=1}^{m} H_k = m H_m - n \]

difficult sum

and book sum = our sum

\[ \sum_{1 \leq k \leq m} \frac{1}{k} = \sum_{1 \leq k \leq n-1} \frac{1}{k} \]

\[ 1 \leq j \leq n-k \]

\[ n \neq n-1 \]

in book remarks
**Problem:** Find closed formula for

\[ \Box_n = \sum_{k=0}^{m} k^2 \]

**Method 0:** Look it up

p. 72 of *CRC Standard Mathematical Tables*

\[ \Box_n = \frac{n(n+1)(2n+1)}{6} \quad n \geq 0 \]

**Other reference:**
- *Handbook of Math. Functions*: Abramowitz, Stegun
- *Handbook of Integer Sequences*: Sloane
Method 1

**Guess the answer and prove by Math Induction**

**Guess:**

\[
\begin{align*}
\sum_{m} m(m+\frac{1}{2})(m+1) &= \frac{m}{3} \\
\sum_{k=0}^{m} k^2 &= m(m+\frac{1}{2})(m+1)
\end{align*}
\]

Re-write as

\[
\begin{align*}
\sum_{0}^{m} &= 0, \\
\sum_{m}^{m} &= \sum_{m-1}^{m} + n^2
\end{align*}
\]

Use Induction Assumption \( n := n-1 \)

\[
3 \sum_{m} = (m-1)(n-1+\frac{1}{2})n + 3n^2
\]

\[
= (n-1)(n-\frac{1}{2})n + 3n^2
\]

\[
= n^3 - \frac{3}{2}n^2 + \frac{1}{2}n + 3n^2
\]

\[
= n^3 + \frac{3}{2}n^2 + \frac{1}{2}n
\]

\[
= m(n^2 + \frac{3}{2}n + \frac{1}{2})
\]

\[
= m(n+\frac{1}{2})(n+1)
\]
METHOD 2: PERTURB THE SUM

\[ \sum_{k=0}^{m} (k+1)^2 = \sum_{k=0}^{m} (k^2 + 2k + 1) = \sum_{k=0}^{m} k^2 + 2 \sum_{k=0}^{m} k + \sum_{k=0}^{m} 1 \]

\[ = \sum_{k=0}^{m} k^2 + 2 \sum_{k=0}^{m} k + m \]

NICE CALCULATION BUT NO RESULT FOR \( \sum_{k=0}^{m} k \)!

NEVERTHELESS WE GET SOMEthing:

\[ (n+1)^2 = 2 \sum_{k=0}^{m} k + (n+1) \]

\[ 2 \sum_{k=0}^{m} k = (n+1)^2 - (n+1) \]

BONUS!
2 \sum_{k=0}^{m} k = (m+1)(n+1-1)

\sum_{k=0}^{m} k = \frac{m(m+1)}{2}

Observation: Use \( k^3 \)

\sum_{k=0}^{m} k^2 = \frac{m(m+1)(2m+1)}{6}

\square_n = \sum_{k=0}^{m} k^3 

Use perturbation for \( \square_n \) to get \( \square \) as we did for \( \sum_{k=0}^{m} k \)

WE EVALUATE (as before)

\square_n + (n+1)^3 = \sum_{k=0}^{m} (k+1)^3

= \sum_{k=0}^{m} (k^3 + 3k^2 + 3k + 1)

\( (a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3 \)
\[ \sum_{k=0}^{n} k^3 + (n+1)^3 = \sum_{k=0}^{n} k^2 + 3 \sum_{k=0}^{n} k + \sum_{k=0}^{n} 1 \]

\[ = \sum_{k=0}^{n} + 3 \sum_{k=0}^{n} k + 3 \cdot \frac{n(n+1)}{2} + (n+1) \]

\[ = \sum_{k=0}^{n} + 3 \cdot \frac{n(n+1)}{2} + (n+1) \]

\[ (n+1)^3 = 3 \sum_{k=0}^{n} + 3 \cdot \frac{n(n+1)}{2} + (n+1) \]

\[ 3 \sum_{k=0}^{n} = (n+1)^3 - 3 \cdot \frac{n(n+1)}{2} - (n+1) \]

\[ 3 \sum_{k=0}^{n} = (n+1)(n+1)^2 - \frac{3}{2} n(n+1) - 1 \]

\[ = (n+1)(n^2 + 2n + 1 - \frac{3}{2} n - 1) \]

\[ = (n+1)(n^2 + \frac{1}{2} n) \]

\[ = (n+1)(n+\frac{1}{2})n \]

End.
Method 3: Build a repertoire

\[ D = \sum_{m} k^2 \]

Generalize:
1. \[ R_0 = \alpha \]
2. \[ R_m = R_{n-1} + \beta + \gamma n + \delta n^2 \]

Used to evaluate \[ \sum_{k=0}^{m} (a+bk) \]

\[ 2a = \beta = a \quad \gamma = b \]

Observe: \[ \delta = 0 \] we get 1

General form of C formula is

\[ R_m = A(m) \alpha + B(m) \beta + C(m) \gamma + D(m) \delta \]

Observe: When \[ \delta = 0 \] we get 1

\[ A(n) = 1, \quad B(n) = n \]

\[ C(m) = \frac{(n^2+n)}{2} \]

Already done before for \[ \sum (a+bk) \]
\[ \sum k^2 \quad \text{generalize} \]
\[ \sum (\alpha + bk^2) \]
\[ R_0 = 2 \]
\[ R_m = R_{m-1} + \beta + \delta n + \delta n^2 \]

get for \[ \beta = a \quad \delta = b \]
\[ \sigma = 0 \]
\[ S_n = \sum (\alpha + bk^2) \]
\[ S_0 = a \]
\[ S_m = S_{n-1} + a + bk^2 \]
GENERAL CF FORMULA BECOMES

\[ R_n = \alpha + m\beta + \frac{(n^2 + n)}{2} \gamma + D(m) \delta \]

WE NEED to evaluate \( D(m) \n^3 \)

PUT \( R_m = n^3 \), for all \( m \)

to evaluate \( \alpha, \beta, \gamma, \delta \) (if it exists)

OUR RECURRENCE \( 2 \) becomes

\[ R_0 = \alpha \]
\[ R_n = R_{n-1} + \beta + \gamma n + \delta n^2 \]

becomes \( R_0 = 0 \) i.e \( \alpha = 0 \)

\[ n^3 = (n-1)^3 + \beta + \gamma n + \delta n^2 \]
\[ n^3 = n^3 - 3n^2 + 3n - 1 + \beta + \gamma n + \delta n^2 \]
\[ 0 = n^2(\delta - 3) + n(\gamma + 3) + (\beta - 1) \]

ALL \( n \)!

ONLY WHEN

\[ \delta = 3, \quad \gamma = -3, \quad \beta = 1 \]
Our CF is

\[ R_m = \alpha + m \beta + \frac{n^2 + n}{2} \gamma + D(m) \delta \]

For \( R_n = n^3 \) and \( \alpha = 0, \beta = 1, \gamma = -3, \delta = 3 \)

It becomes

\[ n^3 = 0 + n - \frac{3}{2} (n^2 + n) + 3D(m) \]

\[ 3D(m) = n^3 - n + \frac{3}{2} (n^2 + n) \]

\[ = n^3 + \frac{3}{2} n^2 + \frac{3}{2} n - n \]

\[ = n^3 + \frac{3}{2} n^2 + \frac{1}{2} n \]

\[ = n(n^2 + \frac{3}{2} n + \frac{1}{2}) = n(n + \frac{1}{2})(n + 1) \]

\[ D(m) = \frac{n(n + \frac{1}{2})(n + 1)}{3} \]

Closed formula

\[ R_m = \alpha + m \beta + \frac{n^2 + n}{2} \gamma + \frac{n(n + \frac{1}{2})(n + 1)}{3} \delta \]
our sum
\[ \sum_{k=0}^{n} k^2 = \]

is a special case of
\[ R_0 = d \]
\[ R_m = R_{m-1} + \beta + \sigma n + \delta n^2 \]

for \( d = 0, \beta = 0, \sigma = 0, \delta = 1 \)

and closed formula
\[ R_m = d + n \beta + \frac{n^2 + n}{2} \sigma + \frac{n(n+1)(n+2)}{3} \delta \]

becomes
\[ R_m = \sum_{n} = \frac{n(n+1)(n+2)}{3} \]

i.e.
\[ \sum_{k=0}^{n} k^2 = \frac{n(n+1)(n+2)}{3} \]
**METHOD 4:**

REPLACE SUMS BY INTEGRALS

\[ f(x) = x^2 \]

**Equation:**

\[ \int_0^m x^2 \, dx = \frac{x^3}{3} \bigg|_0^m = \frac{m^3}{3} \]

**Area under the curve:**

\[ D_n = \frac{m^3}{3} + E_n \]

**Approximation:**

\[ D_n \approx \frac{m^3}{3} \]
\[ E_m = O_m - \frac{1}{3} n^3 \]

Want recursive formula for \( E_m \)

\[ E_{n-1} = O_{n-1} - \frac{1}{3} (n-1)^3 \]

\[ = O_{n-1} - \frac{1}{3} (n^3 - 3n^2 + 3n - 1) \]

\[ = D_{n-1} - \frac{1}{3} n^3 + n^2 - \frac{2}{3} n + \frac{1}{3} \]

Knowing:

\[ E_m = O_{n-1} + n^2 - \frac{1}{3} n^3 \]

\[ = D + n^2 - \frac{1}{3} n^3 + \frac{1}{3} - n \]

\[ - \frac{1}{3} + n \]

\[ E_0 = 0 \]

Closed formula:

\[ E_m = \sum_{k=1}^{m} (k - \frac{1}{3}) \]
E_{m} = \sum_{k=1}^{n} (k - \frac{1}{3}) = \sum_{k=1}^{n} k - \frac{1}{3} \sum_{k=1}^{n} 1

= \frac{n(n+1)}{2} - \frac{1}{3} n

= E_{m} + \frac{n^3}{3}

= \frac{n(n+1)}{2} - \frac{1}{3} n + \frac{n^3}{3}

= \frac{3n^2 + 3n - 2n + 2n^3}{6}

= \frac{2n^3 + 3n^3 + n}{6}

= \frac{n^3 + \frac{1}{2} n^2 + n^2 + n}{3}

= \frac{n^3 + \frac{3}{2} n^2 + \frac{3}{2} n + \frac{1}{2} n}{3}

CHECK

D_{m} = \frac{n(n+\frac{1}{2})(n+1)}{3} = \frac{n^3 + \frac{1}{2} n^2 + n^2 + n}{3} = \frac{n^3 + \frac{3}{2} n^2 + \frac{3}{2} n + \frac{1}{2} n}{3}
2.6 FINITE AND INFINITE CALCULUS

INFINITE CALCULUS:

Derivative operator $D$

$$D f(x) = \lim_{{h \to 0}} \frac{{f(x+h) - f(x)}}{h} = f'(x)$$

Defined for some functions $f: \mathbb{R} \to \mathbb{R}$

called differentiable

$D$ is called operator because it is a function that transforms some functions into different functions

$D: \mathbb{R} \to \mathbb{R}^\mathbb{R}$ not onto
Finite Calculus

Difference operator $\Delta$

$\Delta f(x) = f(x+1) - f(x)$

$\Delta$ is defined for all functions $f$

$f: \mathbb{R} \to \mathbb{R}$

$\Delta$ transforms any function $f$ into another function $g(x) = f(x+1) - f(x)$

$\Delta: \mathbb{R} \to \mathbb{R}$

not onto
Example

\[ f(x) = x^m \]

\[ f : \mathbb{R} \to \mathbb{R} \]

\[ D(f(x)) = mx^{m-1} \]

\[ D(x^m) = m \cdot x^{m-1} \]

\[ f(x) = x^3 \]

\[ D(x^3) = 3x^2 \]

WHAT ABOUT \( \Delta \)?

\[ \Delta(x^3) = (x+1)^3 - x^3 = 3x^2 + 3x + 1, \text{ so} \]

\[ \Delta \neq D \]

\[ \boxed{\text{YES}} \]

Q: Is there a function \( f \) for which \( \Delta f = Df \)?

But there is a "new power" of \( x \), which transforms as nicely under \( \Delta \), as \( x^m \) does under \( D \).
**Falling Factorial Power**

**Definition**

\[ f(x) = x^m = x(x-1)(x-2) \ldots (x-m+1) \]

\[ f: \mathbb{R} \rightarrow \mathbb{R} \]

We also define a

**Rising Factorial Power**

\[ f(x) = x^m = x(x+1)(x+2) \ldots (x+m-1) \]

\[ f: \mathbb{N} \rightarrow \mathbb{N} \]

**Evaluate**

\[ n^m = n(n-1)(n-2) \ldots (n-m+1) = n! \]

**Evaluate**

\[ 1^n = 1 \cdot 2 \cdot \ldots \cdot (1+n-1) = n! \]
HOW TO DEFINE CASE $m=0$

$X^0 = x(x-1) \ldots (x+1)$

$X^0 = x(x+1) \ldots (x-1)$

PRODUCT OF NO FACTORS

$= 1$

SUM OF NO FACTORS

$= 0$
We define

\[ X = X^0 = 1 \]

PRODUCT OF NO FACTORS

\[ 0! = 1 \]

\[ 1^0 = 0! = 1 \]
\[ 0^0 = 0! = 1 \]

We proved:

\[ n^! = n = 1 \quad \text{for any} \quad m \geq 0 \]

EVALUATE

\[ \Delta \left( x^\overline{m} \right) = (x+1)^\overline{m} - x^\overline{m} \]

TO PROVE:

\[ \Delta \left( x^\overline{m} \right) = m \overline{m} \overline{m-1} \]

\[ D(x^m) = mx^m \]

\[ D \text{ "behaves" like } D \text{ on } x^m \]
Evaluate
\[ \frac{m}{(x+1)^m} = (x+1) \times (x-1) \times \cdots \times (x+1-m+1) \]

= \[(x+1)(x-1) \cdots (x-m+2)\]

Evaluate
\[ \frac{m}{x^m} = x(x-1) \cdots (x-m+2)(x-m+1) \]

Evaluate
\[ \Delta (x^m) = (x+1)^m - x^m \]

= \[(x+1)x(x-1) \cdots (x-m+2) - x(x-1) \cdots \frac{(x-m+2)}{(x-m+1)}\]

= \[x(x-1) \cdots (x-m+2)(x+1 - (x-m+1))\]

= \[x(x-1) \cdots (x-m+2) \cdot m\]

= \[m \cdot \frac{m-1}{x} \]

end.