CHAPTER 2: SUMS

INTRODUCTION:

1. Sequences and sums of sequences

2. Definition (of a sequence of elements of a set $A$)

A sequence of elements of a set $A$ is any function $f$ of any natural number $n$,

\[ f : N \rightarrow A \]

Any \( f(n) = a_n \) is called the \( n \)-th term of a sequence $f$.
Example of a sequence of real numbers

\[ f : \mathbb{N} \rightarrow \mathbb{R} \]

\[ f(n) = n + \sqrt{n} \]

\[ a_n = n + \sqrt{n} \]

or

\[ a_n = n + \sqrt{n} \]

\[ a_0 = 0, \quad a_1 = 1 + 1 = 2, \quad a_2 = 2 + \sqrt{2} \ldots \]

\[ 0, 2, 2 + \sqrt{2}, 3 + \sqrt{3}, \ldots \quad \text{natural numbers, \ldots} \]

**Observation 1**

Sequence is always infinite (countably infinite) as by definition, domain of \( f \) is a set \( \mathbb{N} \) of natural numbers.

\[ A \cap B = \{1, 18\} \]

**Observation 2**

\[ \mathbb{N} \sim \mathbb{N} - \{0\} = \mathbb{N}^+ \]

\[ \mathbb{N} \sim \mathbb{N} - \{K\}, \text{K any finite subset of } \mathbb{N} \]

\[ \mathbb{N} \sim \mathbb{Z}, \mathbb{N} \sim \text{odd}, \mathbb{N} \sim \text{even} \]
In general: a set $T$ is only COUNTABLY INFINITE iff $|T| = |N|$ i.e. there is a function $f: N \xrightarrow{1:1} \text{onto} T \quad (T \sim N)$

**Observation 3**

We can choose a set of indexes of a sequence from any COUNTABLY INFINITE set $T$, not only $N$.

**In our book:** $T = N - \{0\} = N^+$

i.e. sequences "start" with $n = 1$.

**General form of a sequence:**

**Finite sequence:** any $f$ $f: K \rightarrow A$, where $|K| = n \in N$ or $K$ - finite subset of $N$. 
Finite sequence (2)

\[ f: \{1, \ldots, n\} \rightarrow A, \quad m \in \mathbb{N} \]

\[ f(m) = a_m \]

\[ a_1, a_2, \ldots, a_m \]

\[ \{a_k\}_{k=1}^{n} \quad k=1, 2, \ldots, n \]

**Empty sequence:** case \( m = 0 \)

\[ f(\emptyset) = e \]

**Domain of the sequence**

Formula:

\[ a_m = \frac{n}{(n-2)(n-5)} \]

Domain of \( f \) is

\[ T = \{1, 2, 3, 4\} \]

\[ f: N \setminus \{-2, 5\} \rightarrow \mathbb{R} \]

\[ f(m) = a_m \]

\[ T \subseteq \mathbb{Z} \]
Sums of elements of a sequence of rational \#s.

In Chapter 2, we consider only finite sums of consecutive elements of a sequence \{a_n\} of rational numbers.

**Definition**

Given a sequence \( f : \mathbb{N}^+ \rightarrow \mathbb{R} \) of rational numbers,

\[ f(n) = a_n \]

We write

\[ \sum_{k=1}^{n} a_k = a_1 + a_2 + \ldots + a_n \]

\[ \sum_{k=1}^{m} a_k = \sum a_k = \sum a_k = \sum a_k \]

for \( k = \{1, \ldots, n\} \).
Given a sequence of numbers:

\[ f : \mathbb{N}^+ \rightarrow \mathbb{R} \]

or

\[ a_1, a_2, \ldots, a_n, \ldots \]

\[ a_k \in \mathbb{R} \]

Full definition:

SHORTHAND:

we sometimes need to evaluate a sum of some sub-sequence of \( \{a_n\} \) only.

For example:

1. Sum up only each second term of \( \{a_n\} \)
   (i.e. \( n \) \text{ even})

   We write it two ways:

   \[ \sum_{1 \leq k \leq 2n} a_k \quad \text{Ke EVEN} \]
   
   \[ \text{p(k) summation property} \]

   \[ \sum_{k=1}^{n} a_{2k} = a_2 + a_4 + \ldots + a_{2n} \]

   \[ \text{subsequence property} \]
We use notation

\[ \sum_{k \in K} a_k = \sum_{P(k)} a_k \]

for

\[ K = \{ m \in \mathbb{N} : P(m) \} \]

\( P(m) \) is a certain formula (predicate) defining our restriction on \( m \).

We assume

1. \( K \) is defined; i.e., \( P(m) = T \) or \( F \) is decidable

2. \( K \) is finite
   (we consider only finite sums for a moment)
Example 1

Let

\[ P(m) = (1 \leq n < 100) \land (n \text{ ODD}) \]

\[ P(m) \text{ is a formula (open) defining all ODD numbers between 1 and 99 (included)} \]

\[ K = \{ n \in \mathbb{N} : P(m) \} \]

\[ = \{ 1, 3, 5, \ldots, 99 \} \]

\[ = \{ n \in \mathbb{N} : 1 \leq n \leq 99 \} \]

\[ = \{ 1, 3, \ldots, (2n+1) \} \quad \text{for} \quad 0 \leq n \leq 49 \]

\[ \sum_{P(m)} a_m = \sum_{K \in K} a_k \]

\[ = \sum_{n=0}^{49} a_{(2n+1)} = a_1 + a_3 + \ldots + a_{99} \]
Example 2

Let

\[ P(m) = (1 \leq n < 100) \]

\( P(m) \) is a predicate defining all men between 1 and 99 included.

In this case

\[ \sum a_m = \sum_{K \in K} a_K = \sum_{K=1}^{99} a_K \]

\[ = a_1 + a_2 + a_3 + \ldots + a_{99} \]

Example 3

Let

\[ a_m = (2n+1)^2 \]

\[ P(m) = (1 \leq n < 100) \]

Evaluate:

\[ \sum_{P(m)} \frac{(2n+1)^2}{K=1}^{99} \]
(From Kenneth Iverson's programming language APL)

\[
[P(x)] = \begin{cases} 
1 & P(x) \text{ true} \\
0 & P(x) \text{ false} 
\end{cases}
\]

Example

\[
[p \text{ prime}] = \begin{cases} 
1 & p \text{ is prime} \\
0 & p \text{ is not prime} 
\end{cases}
\]

We write

\[
\sum a_k = \sum a_k [P(k)]
\]

\[K = \{k \in \mathbb{N} : P(k)\}\]
Example

\[
\sum_{p \text{ prime}} \left[ \left( p \leq n \right) \right] \frac{1}{p}
\]

Property

\( P(x) \) is \( P_1(x) \cap P_2(x) \)

for \( P_1(x) : x \text{ is prime} \)

\( P_2(x) : x \leq n \), for \( n \in \mathbb{N} \)

\( P(x) \) says: \( x \) is prime and \( x \in \mathbb{N} \)

\( \Sigma \) means: we sum over all \( \mathbb{P} \) \( p \) that are prime and \( p \leq n \)

CASE \( n = 0 \) \( P(x) \) is **FALSE**

PRIMES are natural numbers 3 2 etc..
Book notation

$p \leq n$ for $p \leq m$ \text{ whenever } n \in \mathbb{N}

where $\mathbb{N}$ denotes a natural number. IT IS NOT CORRECT

$\mathbb{N}$ always denotes a set of Natural numbers

I will use $p \leq m$ \text{ whenever } n \in \mathbb{N}

When you read the book now and later, pay attention it happens often

$m \leq K$ \text{ means that } m \leq K \text{ for some } K\in \mathbb{N}$

usually \textbf{CAPITAL LETTERS DENOTE SETS.}
1. Authors never define a sequence \( \{a_n\} \) for \( \sum a_k \).

2. They say often "\( a_k \) is defined/not defined for all set of integers".

   IT MEANS they admit finite sequences

\[ f: K \rightarrow A \quad f(k) = a_k \]

for \( K \) finite subset of \( \mathbb{Z} \).
\[ \sum a_k = \sum a_k = \sum_{k \in K} [p(k)] a_k \]

where

\[ K = \{ k \in \mathbb{Z} : p(k) \} \]

and \( K \) is FINITE.

You can put

\[ K = \{ k \in \mathbb{R} : p(k) \} \text{ or } \begin{cases} 1 & \text{if } p(k) \text{ true} \\ 0 & \text{if } p(k) \text{ false} \end{cases} \]

\( K \) finite

\[ K = \{ k \in \mathbb{N} : p(k) \} \]

and \( K \) is FINITE

This is usual case.
**Example 4**

\[
\sum_{k=1}^{n} a_k = \sum_{k=0}^{n-1} a_{k+1} \quad \text{changes limits}
\]

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**Sums and Recurrences**

Observation, for any \( n \in \mathbb{N} \):

\[
\sum_{k=1}^{m+1} a_k = \sum_{k=1}^{n} a_k + a_{m+1}
\]

**Case \( n = 0 \)**

\[
\sum_{k=1}^{1} a_k = a_1, \quad \sum_{k=1}^{0} a_k = 0
\]

Define:

\[
\sum_{k=a}^{b} a_k = 0 \quad \text{when } b < a
\]

In general, when sum is undefined, we put it to 0.
Sums and Recurrences

Observation:

\[
\sum_{k=0}^{m} a_k = \sum_{k=0}^{n-1} a_{k-1} + a_m
\]

\[n = 0\]

\[
\sum_{k=0}^{0} a_k = a_0
\]

\[
\sum_{k=0}^{1} a_k = 0
\]

\[
\sum_{k=0}^{n} a_k = a_0 = 0 + a_0 = a_0
\]

\[n = 1\]

\[
\sum_{k=0}^{1} a_k = a_0 + a_1,
\]

\[
\sum_{k=0}^{0} a_k = a_0, \quad a_n = a_1,
\]

\[
a_0 + a_1 = a_0 + a_1,
\]
We know
\[ \sum_{k=0}^{m} a_k = \sum_{k=0}^{h-1} a_k + a_m \]

Denote
\[ S_m = \sum_{k=0}^{m} a_k \]

We re-write \(*\) as

Recursive form (Recurrence)

\[ S_0 = a_0 \]
\[ S_m = S_{h-1} + a_m \]
for \( n > 0 \)

We can use techniques from CHAPTER 1 to evaluate (if possible) closed formulas for sums.
Example

Given a sequence
\[ f: \mathbb{N} \rightarrow \mathbb{R} \quad f(n) = a_n \]
\[ a_n = a + b \cdot n \]

Parameters
\[ a, b \in \mathbb{R} \]

Find a closed formula for
\[ S(m) = \sum_{k=0}^{m} a_k = \sum_{k=0}^{m} (a + b \cdot k) \]

The recurrence form of \( S(m) \)

\[ S_0 = a \]
\[ S_m = S_{m-1} + (a + b \cdot n) \]

We want to find a CF for this recurrence formula.
WE CONSIDER A MORE GENERAL CASE

\[ R_0 = \lambda \]
\[ R_m = R_{m-1} + (\beta + \gamma n) \]

RS is a case of R for
\[ a = a \]
\[ \beta = a \]
\[ \gamma = b \]

and look for CF

STEP 1: Evaluate few terms
\[ R_0 = \lambda \]
\[ R_1 = \lambda + \beta + \gamma \]
\[ R_2 = \lambda + \beta + \gamma + (\beta + 2\gamma) = \lambda + 2\beta + 3\gamma \]
\[ R_3 = \lambda + 2\beta + 3\gamma + (\beta + 3\gamma) = \lambda + 3\beta + 6\gamma \]

STEP 2: Observation:
In general,\[ R_m = A(m)\alpha + B(m)\beta + C(m)\gamma \]

**Goal:** Find $A(m)$, $B(m)$, $C(m)$ and this proves $R = CF$

**Method:** Repertoire method

**Step 1:** Set $R_n = 1$, all $n \in \mathbb{N}$

i.e. $R(m) = 1 = R_m$ is a constant function (if possible)

Use $R_0 = L$, $R_m = R_{m-1} + (\beta + \gamma \cdot n)$

$R_0 = 1$ gives $d = 1$

$R_m = R_{m-1} + (\beta + \gamma \cdot n)$ gives $1 = 1 + (\beta + \gamma \cdot n)$ for all $n \in \mathbb{N}$

Evaluate:
We get

\[ 0 = \beta + \delta \cdot n \]  
for all \( n \in \mathbb{N} \)

This is possible only when

\[ \beta = \delta = 0 \]

We obtained

\[ \alpha = 1, \quad \beta = 0, \quad \delta = 0 \]

And our closed formula

\[ R_m = A(m) \delta + B(m) \beta + C(m) \delta \]

becomes

\[ R_m = A(m) \delta = A(m) = 1, \quad \text{all } m \in \mathbb{N} \]

We proved

**FACT 1**  
\[ A(m) = 1, \quad \text{for all } m \in \mathbb{N} \]
STEP 2

We set $R_n$ momentarily

$$R_m = n, \text{ all } n$$

and find $\alpha, \beta, \gamma$ if exist.

$R_0 = \alpha, R_n = R_{n-1} + (\beta + \gamma n)$

$R_0 = \alpha = 0$ gives $\alpha = 0$

$R_m = R_{m-1} + (\beta + \gamma n)$ becomes

$\alpha = (n-1) + \beta + \gamma n$

$1 = \beta + \gamma n$ for all $n$

possible only when

$\beta = 1, \gamma = 0$

We evaluate $CF$ for

$\alpha = 0, \beta = 1, \gamma = 0$
\[ R_m = A(m) \alpha + B(m) \beta + C(m) \gamma \]

becomes for \( \alpha = 0, \beta = 1, \gamma = 0 \) and \( R_m = n, \forall m \in \mathbb{N} \)

\[ n = A(m) \cdot 0 + B(m) \cdot 1 + C(m) \cdot 0 \]

\( \text{FACT2} \)

\[ B(m) = m \quad \text{for all } m \in \mathbb{N} \]

\( \text{STEP 3} \)

We set \( R_m = m^2 \), \( \forall m \in \mathbb{N} \)

and find \( \alpha, \beta, \gamma \), if it exist.

\( A(m) \) is a function \( A : \mathbb{N} \rightarrow R \)

\( B(m) \) is a function \( B : \mathbb{N} \rightarrow R \)
\[ R_0 = \alpha, \quad R_n = R_{n-1} + (\beta + \delta m) \]
\[ R_m = m^2, \text{ all } m \]

\[ R_0 = \alpha = 0^2 \quad \alpha = 0 \]

\[ n^2 = (n-1)^2 + \beta + \delta m, \quad \text{all } m \in \mathbb{N} \]
\[ n^2 = n^2 - 2n + 1 + \beta + \delta m, \quad \text{all } m \in \mathbb{N} \]
\[ 0 = -2n + 1 + \beta + \delta m, \quad \text{all } m \in \mathbb{N} \]
\[ 0 = (1 + \beta) + m(\delta - 2) \]

If
\[ 1 + \beta = 0 \quad \delta - 2 = 0 \]
\[ \beta = -1 \quad \delta = 2 \]

We get:
\[ \alpha = 0, \beta = -1, \delta = 2 \]

and

calculate CF
\[ R_n = A(m) \alpha + B(m) \beta + C(m) \delta \]

for \( R_n = n^2 \), \( \alpha = 0 \), \( \beta = -1 \), \( \delta = 2 \) all \( n \in \mathbb{N} \)

\[ n^2 = -B(m) + 2C(m) \quad \text{all } n \in \mathbb{N} \]

We know (FACT2) that:
\[ B(m) = m, \quad \text{all } n \in \mathbb{N} \]

\[ n^2 = -n + 2C(m) \quad \text{all } n \in \mathbb{N} \]

FACT 3:
\[ \frac{n^2 + n}{2} = C(m) \]

FACT 1 + 2 + 3:
\[ A(m) + 1 \]

FACT 1:
\[ R_m = \alpha + m \beta + \left( \frac{n^2 + n}{2} \right) \delta \]
Go back to

\[ S_m = \sum_{k=0}^{n} (a+bk) \]

\[ S_m = R_n \quad \text{for} \quad d = a, \beta = \alpha \]

\[ R_n = d + m\beta + \left( \frac{m^2 + n}{2} \right) \gamma \]

\[ S_m = a + ma + \left( \frac{n^2 + n}{2} \right) b \]

\[ S_m = (m+1)a + \frac{m(m+1)}{2} b \]

and we evaluated

\[ \sum_{k=0}^{n} (a+bk) = (n+1)a + \frac{m(m+1)}{2} b \]
OF course we can do it by a "simpler method"

\[ \sum_{k=0}^{n} (a + bk) = \]

\[ \text{Properties of summation to be listed next} \]

\[ p_1 = \sum_{k=0}^{n} a + \sum_{k=0}^{n} b \cdot k \]

\[ = (n+1)a + b \sum_{k=0}^{n} k \]

\[ = (n+1) + \frac{n(n+1)}{2} b \]

\[ a_n = a, \text{ all } n \]

\[ \sum_{k=0}^{n} a_n = \sum_{k=0}^{n} a = a + \cdots + a = (n+1)a \]
SUMMATION PROPERTIES

**P1**
\[ \sum_{k \in K} c \cdot a_k = c \sum_{k \in K} a_k \]
DISTRIBUTIVE LAW

**P2**
\[ \sum_{k \in K} (a_k + b_k) = \sum_{k \in K} a_k + \sum_{k \in K} b_k \]
ASSOCIATIVE LAW

**P3**
\[ \sum_{k \in K} a_k = \sum_{\pi(k) \in K} a_{\pi(k)} \]
COMMUTATIVE LAW

\[ \pi(k) - \text{any permutation of elements of } K \]
K - FINITE SUBSET OF INTEGERS

(p. 30)
**SUM OF GEOMETRIC SEQUENCE**

**GEOMETRIC SUM**

**DEF**

\[ f : \mathbb{N} \to \mathbb{R} \text{ is geometric } \iff \ \forall n \in \mathbb{N} \left( \frac{a_{n+1}}{a_n} = q \right) \]

We prove **FACT**

\[ a_n = a_0 q^n \quad \text{for geometric sequence for all } n \in \mathbb{N} \]

**GEOMETRIC SUM** = **SUM OF A GEOMETRIC SEQUENCE**

\[ S_m = \sum_{k=0}^{m} a_0 q^k = \frac{a_0 (1 - q^{m+1})}{1 - q} \]

\[ S_m = a_0 + a_0 q + \ldots + a_0 q^{m+1} \]

\[ q \cdot S_m = a_0 q + a_0 q^2 + \ldots + a_0 q^{m+1} \]

\[ S_m (1 - q) = a_0 - a_0 q^{m+1} \]
**Geometric Sum**

\[ S_m = \sum_{k=0}^{m} a_0 q^k = a_0 \frac{q^{m+1} - 1}{q-1} \]

**Example**

\[ S_m = \sum_{k=0}^{m} 2^{-k} = \sum_{k=0}^{m} \left(\frac{1}{2}\right)^k \]

\[ a_0 = 1 \quad q = \frac{1}{2} \]

\[ S_m = \frac{\left(\frac{1}{2}\right)^{m+1} - 1}{-\frac{1}{2}} = 2 - \left(\frac{1}{2}\right)^{m+2} \]

\[ S_m = \sum_{k=1}^{n} 2^{-k} = \sum_{k=1}^{n} \left(\frac{1}{2}\right)^k \]

\[ S_m = a_0 \frac{q^{m+1}}{(q-1)} \]

\[ a_0 = \frac{1}{2} \quad q = \frac{1}{2} \]

\[ S_{m-1} = \frac{\frac{1}{2} \left(\left(\frac{1}{2}\right)^2 - 1\right)}{-\frac{1}{2}} = 1 - \left(\frac{1}{2}\right)^{m+1} \]

**Previous** \( S_m \) minus 1!
**RECURRENCE → SUM → CLOSED FORMULA**

**GOAL**

\[ R \]

\[ T_0 = 0 \]
\[ T_m = 2T_{n-1} + 1 \]

**T**

\[ T : N \rightarrow R \]

Divide by \( 2^n \)

\[ T_{m/2^n} = 0 \]

\[ T_{m/2^n} = 2 \frac{T_{n-1}}{2^m} + \frac{1}{2^m} = \frac{T_{n-1}}{2^{m-1}} + \frac{1}{2^m} \]

**DENOTE**

\[ S_m = T_{m/2^n} \]

**IT**

\[ S_0 = 0 \]
\[ S_m = S_{m-1} + \frac{1}{2^n} \]

\[ S_m = \sum_{k=0}^{m} \frac{1}{2^k} \]

**Means**

\[ T = S \]
\[ T(0) = S(0) = 0 \]

\[ T(n) = S(n) \]

\[ T(n) = S(n) = 5^n \]
We know (as $S_m$ is geometric): 

$$S_m = 1 - \frac{1}{2^m}$$

We use 

$$S_m = \frac{T_m}{2^m}$$

$$T_m = 2^m S_m$$

to get a closed formula for $T_m$

$$T_m = 2^m \left(1 - \frac{1}{2^m}\right) = 2^m - 1$$