CHAPTER 5  pp 153 - 243

BINOMIAL COEFFICIENTS

DEF 1
\[
\binom{n}{k} = \frac{n(n-1)...(n-k+1)}{k(k-1)...2\cdot1} = \frac{n!}{k!(n-k)!}
\]

\[
\binom{n}{k} = \frac{n!}{k!(n-k)!}
\]

FOR \( k > 0, n \in \mathbb{N} \)

K \in \mathbb{N}

COMBINATORIAL INTERPRETATION

\[
\binom{n}{k} \quad \text{"n choose k"}
\]

\[
\binom{n}{k}
\]

denotes a NUMBER of ways to choose a \( k \)-ELEMENT SUBSET from an \( n \)-ELEMENT SET.
**Proof of Combinatorial Statement**

\[
\binom{m}{k} = \frac{m^k}{k!}
\]

is the number of ways to choose \( k \)-element subsets from \( m \)-element set.

**Step 1**

Find the number of \( k \)-element 1-1 sequences formed out of \( m \)-element set.

**Remark 1**
all sequences of length \( k \) from \( m \)-element set are all functions 
\[ f : \{1, \ldots, k\} \rightarrow \{a_1, \ldots, a_m\} \]
and there are \( \binom{m}{k} \) of them. 
We need 1-1 sequences only.

**Remark 2**

**Def** permutation of \( A = \{a_1, \ldots, a_m\}, |A| = m \) is any function 
\[ f : A \rightarrow A \]
we prove:
\[ |A| = m, \text{ then } \# \text{ of permutations of } A \text{ is } m! \]
**Proof of Remark 2**

If $|A| = n$, then there are $n!$ functions $f : A \rightarrow A$, $n \geq 1$.

By induction over $|A| = m$.

- **$n = 1$**
  - Assume that $B \subseteq A$ (any subset of $A$) and $|B| = n - 1$.
  - Then there is $(n-1)!$ functions that map $f : B \rightarrow B$.
  - We count $f : A \rightarrow A$ as follows.

- Assume $A = \{a_1, a_2, \ldots, a_n\}$.

  - If $f(a_1) = a_1$, then $f : A \rightarrow A$.
    - By inductive assumption, there are $(n-1)!$ such functions.

- **Picture**

  - $A \rightarrow A$
  - $B \rightarrow B$
  - All possible $f : B \rightarrow B$.
  - $f(b) = g(b)$ for $b \in B$.
\begin{equation}
\begin{aligned}
\text{Group 2} \quad & \quad f(a_2) = a_2 \quad \text{fix}_{a_2} \quad \text{fix}_{a_2} + \quad f(b) = g(b) \\
& \quad \text{for } b \in B = \{a_1, a_3 \ldots a_m\} \quad 181 = n - 1 \\
& \quad \text{There are } (m-1)! \text{ functions in } G_2 \\
\text{Group } i \quad & \quad i \geq 2 \\
& \quad f(a_i) = a_i \quad + \quad f(b) = g(b) \\
& \quad \text{for } b \in A - \{a_i\} = B \quad g : B \xrightarrow{\text{fix}} B \\
& \quad \text{There are } (n-1)! \text{ functions}
\end{aligned}
\end{equation}

There are \( n \) groups; all together there are \( n \cdot (m-1)! \text{ functions } f : A \xrightarrow{\text{fix}} \)

\( m! = n(m-1)! \) \text{ end of proof of Remark 2}

In particular \( A = \{a_1, \ldots a_k\} \)

There are \( k! \) 1-1 sequences of length \( k \) out of \( A \).
Let |A| = n be an n-element set. We construct all 1-1 k-element sequences out of elements of A as follows: sequence 1-1, all i1, 0 i < k.

1. a1, b1 - there are n choices, a1 \in A
2. a2, b2 - there are (n-1) choices for b2 \neq b1
3. a3, b3 - there are (n-2) choices for b3 \neq b2 \neq b1, a3 \in A

Induction (really)

6, b6, b7 - there are (n-k+1) choices for b6 \neq b5 \neq b4 \neq \ldots \neq b1

All together we have

m(m+1) \ldots (n-k+1)

possible 1-1 sequences a1, b1, b2, \ldots, b_k
STEP 2 \( \binom{m}{k} \) represents how many are there \( k \)-element SUBSETS

We know that there are \( m(n+1)\ldots(n+k+1) \) sequences 1-1

\[ b_1, b_2, \ldots, b_k \]

\[ \{ b_1, b_2, \ldots, b_k \} \]

\[ \text{subset} \]

\[ \text{SET} \]

\[ \{ \text{SETS} : \{ b_1, b_2, \ldots, b_k \} = \{ b_2, b_3, \ldots, b_k \} \} \]

\[ \text{different sequences} \]

\[ \text{represent the same} \]

\[ \text{SET} \]

How many of all possible "representations" - as many as PERMUTATIONS i.e. \( k! \)

\[ \binom{m}{k} = \frac{\# \text{sequences}}{k!} = \frac{m(m+1)\ldots(m+k+1)}{k!} \]

END
We proved (combinatorial identity) is a number of $k$-element subsets of an $n$-element set.

\[
\binom{n}{k} = \frac{n!}{k!}
\]

$k, n \in \mathbb{N}, k \leq n$

We define $f : \mathbb{R} \rightarrow \mathbb{R}$

\[
f(x) = \frac{k}{x^n} = x^k \cdot x^{(n-k)}
\]

or exactly

\[
f : \mathbb{R} \times \mathbb{N} \rightarrow \mathbb{R}
\]

\[
f(x, k) = \frac{x^k}{x^n} = x^{(k-n)}
\]

We define $f : \mathbb{R} \times \mathbb{Z} \rightarrow \mathbb{R}$

\[
f(x, k) = \begin{cases} \frac{x^k}{k!} & k \geq 0 \\ 0 & k < 0 \end{cases}
\]

or even

\[
f : \mathbb{C} \times \mathbb{Z} \rightarrow \mathbb{C}
\]

c - complex number

\[
f(x, k) = \binom{x}{k} = g(x) - f_x \text{ for any } k \in \mathbb{Z}
\]
**Example**

\[
\binom{x}{k} = \frac{x^k}{k!}
\]

\[
\binom{-1}{3} = \frac{(-1)^3}{3!} = \frac{(-1)(-2)(-3)}{1 \cdot 2 \cdot 3} = -1
\]

\[
\binom{x}{k} = x(x-1) \cdots (x+K+1)
\]

\[
\binom{-1}{0} = 0 \quad (K < 0)
\]

\[
\binom{1}{1} = 1 \quad \text{in general}
\]

\[
\binom{\sqrt{2}}{3} = \frac{\left(\sqrt{2}\right)^3}{3!} = \frac{\sqrt{2} \left(\sqrt{2} - 1\right) \left(\sqrt{2} - 2\right)}{6}
\]

\[
\left(\sqrt{2}\right)^3 = \sqrt{2} \left(\sqrt{2} - 1\right) \left(\sqrt{2} - 2\right)
\]

**No Combinatorial Interpretation!**
We defined $f : \mathbb{R} \times \mathbb{Z} \to \mathbb{R}$

$$f(x, k) = \begin{cases} \frac{x^k}{k!} & k \geq 0 \\ 0 & k < 0 \end{cases}$$

where $x \in \mathbb{R}$, $k \in \mathbb{Z}$

Use $\times$ to denote that $\textit{upper limit is a real number}$

We use $\binom{n}{k} = \frac{n!}{k!(n-k)!}$ for combinatorial interpretation

Book uses $x \in \mathbb{R}$ instead

$$f(x, k) = \begin{cases} \frac{x^k}{k!} & k \geq 0 \\ 0 & k < 0 \end{cases}$$

$f : \mathbb{R} \times \mathbb{Z} \to \mathbb{R}$

SAME FUNCTION
\[ (x) = \frac{x^k}{k!} \quad k \geq 0, \quad (x)_k = 0 \quad k < 0 \]

**Reminder**

1. \( (m)_n = 1 \quad m \in \mathbb{N} \)
2. \( (m)_n = 0 \quad m < 0 \)
3. \( (m)_k = 0 \quad \) for \( k > n \), \( k > 0 \)

**Remark**

we may remove restrictions \( k > 0 \) in

\[ (x)_k = \frac{x^k}{k!} \]

i.e. \( (x)_k = 0 \quad k < 0 \)

Let's look at

**Symmetry Identity**
Consider \( f : \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{N} \) \( \quad 0 \leq k \leq n \)

\[
\binom{n}{k} = \frac{n(n-1) \ldots (n-k+1)}{k!} = \frac{n!}{k! (n-k)!}
\]

\[
= \frac{n!}{(n-(n-k))! (n-k)!} = \binom{n}{n-k}
\]

**Symmetry Property**

\[
\binom{n}{k} = \binom{n}{n-k}
\]

**Combinatorial Interpretation**

\( \binom{n}{k} \) \( k \) chosen elements out of \( n \)

\( \binom{n}{n-k} \) \( n-k \) "unchosen" elements out of \( n \).
Symmetry Property

\[
\binom{m}{k} = \binom{m}{m-k}
\]

Case \( k < 0 \)

\[
\binom{m}{k} = 0, \quad \binom{m}{m-k} = \binom{m}{s} = 0
\]

\( s > m \)

Case \( k > m \)

\[
\binom{m}{k} = 0, \quad \binom{m}{m-k} = \binom{m}{s} = 0
\]

\( s < 0 \)

So we have

\[
\binom{m}{k} = \binom{m}{m-k}
\]

\( m \in \mathbb{N} \), \( k \in \mathbb{Z} \)

\( f : \mathbb{N} \times \mathbb{Z} \rightarrow \mathbb{N} \)

Why can't \( m \in \mathbb{Z} \)?
Check \( \binom{m}{k} = \binom{m}{n-k} \) for \( n = -1 \)

\[
\binom{-1}{k} = \frac{(-1)^k}{k!} = \frac{(-1)(-2)\cdots(-1+k+1)}{k!} \quad \text{K factors}
\]

\[
= \frac{(-1)^k k!}{k!} = \binom{-1}{k}
\]

\[
= \frac{k}{x(x-1)\cdots(x-k+1)}
\]

\[
\binom{-1}{-1-k} = 0 \text{ all } k > 0
\]

By Induction.

\[
\binom{-1}{k} = \binom{-1}{-1-k} \text{ all } k \geq 0.
\]

**Absorption Identity**

\[
\binom{x}{k} = \frac{x}{k} \binom{x-1}{k-1}
\]

\( k \neq 0 \) \( x \in \mathbb{R} \) \( k \in \mathbb{Z}^+ \cup \{0\} \)
Proof of Absorption Identity

1. \( \binom{x}{k} = \frac{x}{k} \binom{x-1}{k-1} \)

2. \( k \binom{x}{k} = x \binom{x-1}{k-1} \)

Proof:

1. Observe \( \frac{k}{x} = x(x-1) \) (Identity)

2. \( x(x-1) \frac{k-1}{x} = x(x-1)(x-2) \ldots (x-1-(k-1)+1) \)

   \[ = x(x-1) \ldots (x-1-k+1+1) \]

   \[ = x(x-1) \ldots (x-k+1) \]

   Evaluate: \( = x(x-1) \ldots (x-k+1) = \frac{x^k}{k} \)

\[ \binom{x}{k} = \frac{x(x-1)\ldots(x-k+1)}{k!} = \frac{x}{k} \binom{x-1}{k-1} \] end
**Absorption Identity**

2.

\[ K \left( \frac{x}{k} \right) = x \left( \frac{x-1}{k-1} \right) \]

\( x \in \mathbb{R} \)
\( k \in \mathbb{Z} \)

**To be proved:**

\[ (x-k) \binom{x}{k} = x \binom{x-1}{k} \]

\( x \in \mathbb{R} \)
\( k \in \mathbb{Z} \)

---

**Proof**

1. **Prove the Assumption case**

   \[ L = (x-k) \binom{x}{k} = (x-k) \binom{x}{x-k} \]

2. **Symmetry**

   \[ L = x \left( \frac{x-1}{x-k-1} \right) \]
We proved identity 3 for a case $\mathbb{X} \in \mathbb{N}, \mathbb{K} \in \mathbb{Z}$.

Now we are going to show that this result extends to $\mathbb{X} \in \mathbb{R}$.

We do it by what is called a polynomial argument.

**Observe:**

\[
(x-1)(\binom{x}{k}) = x \binom{x-1}{k}\]

is equality of two polynomials of degree $(k+1)$.

\[
L(x) = (x-1)\binom{x}{k} = a_k x^k + \ldots + a_0
\]

\[
P(x) = x \binom{x-1}{k} = b_k x^k + \ldots + b_0
\]

\[
\binom{x}{k} = \frac{x(x-1) \ldots (x-k+1)}{k(k-1)!}
\]

polynomial of degree $k$ over $\mathbb{Z}$. 
POLYNOMIALS THEOREMS

**THM 1**

Let \( w(x) = a_m x^n + \ldots + a_0 \quad a_n \neq 0 \)

be a polynomial of the degree \( N \).

The equation

\[ w(x) = 0 \]

has at most \( n \) solutions; i.e.

\[ | \{ x : w(x) = 0 \} | \leq n. \]

**THM 2**

If \( w(x) = a_m x^n + \ldots + a_0 \quad a_n \neq 0 \)

is such that

\[ | \{ x : w(x) = 0 \} | > n \]

then

\[ w(x) = 0 \quad \text{for all } x \in \mathbb{R} \]
BACK TO OUR IDENTITY

\[ L(x) = (x-1)\binom{x}{k} = a_{k+1} x^{k+1} + a_0 \]
\[ P(x) = x \binom{x-1}{k} = b_{k+1} x^{k+1} + \ldots + b_0 \]

3. \[ L(x) = P(x) \]

for all \( x \in \mathbb{R}, k \in \mathbb{K} \)

\[ L(x) - P(x) = 0 \]
\[ \frac{w(x)}{x_0} \]

We have a polynomial \( w(x) = L(x) - P(x) \) of degree \( k+1 \).

\[ \{ x : L(x) - P(x) = 0 \} = \{ x_0 \} \supset k+1 \]

became we proved that \( L(x) = P(x) \) for all \( x \in \mathbb{N} \).

by \textbf{Thm 2} \[ w(x) = L(x) - P(x) = 0 \]

for all \( x \in \mathbb{R} \).

HENCE

\[ L(x) = P(x), \text{ for all } x \in \mathbb{R}, k \in \mathbb{K}. \]
\[(x-k)(x) = x(x-1)\]

\[x \binom{x}{k} - k \binom{x}{k} \cdot x \binom{x-1}{k} \]

\[x \binom{x}{k} - x \binom{x-1}{k} = x \binom{x-1}{k} \]

\[x \binom{x}{k} = x \binom{x-1}{k} + x \binom{x-1}{k-1} \]

\[x \binom{x}{k} = x \left( \binom{x-1}{k} + \binom{x-1}{k-1} \right) \]

\[\binom{x-1}{k} = \frac{(x-1)(x-2) \ldots \ldots (x-1-k+1)}{k!} \]

\[x-1-k+1 = (x-1)(x-2) \ldots \ldots (x-k) \]

\[\binom{x-1}{k-1} = \frac{(x-1)(x-2) \ldots \ldots (x-k+1)}{(k-1)!} \]

\[\frac{(x-1)^k}{(k-1)!} \frac{k}{k} = (1 + \frac{x-k}{k})^{k-1} \]

\[\prod (1 + \frac{x-k}{k}) = \frac{(x-1)^k}{(k-1)!} \]
\[ x(k) = \binom{x}{k-1} \cdot \frac{x}{k} \cdot x \]

\[ \binom{x}{k} = \binom{x-1}{k-1} \cdot \frac{x}{k} \]

\[ k \binom{x}{k} = x \binom{x-1}{k-1} \text{ \quad yes, we prove} \]

+ check case \[ x = 0 \]
**Symmetry Property**

\[
\binom{n}{k} = \binom{n}{n-k}
\]

\(m \in \mathbb{N}, k \in \mathbb{Z}\)

**Absorption Identity**

\[
\binom{n}{k} = \frac{x}{k} \binom{x-1}{k-1}
\]

\(x \in \mathbb{R}, k \in \mathbb{Z} - \{0\}\)

\[
K \binom{x}{k} = x \binom{x-1}{k-1}
\]

\(x \in \mathbb{R}, k \in \mathbb{Z}\)

Prove first for \(x \in \mathbb{N}\), and then extend it via polynomial argument to \(x \in \mathbb{R}\)

\[
\binom{x}{k} = \binom{x-1}{k} + \binom{x-1}{k-1}
\]

\(x \in \mathbb{R}, k \in \mathbb{Z}\)

\[
\binom{n}{k} = \frac{n}{k!} = \frac{n(n-1)...(n-k+1)}{k!}
\]

\(k > 0, k \in \mathbb{N}, n, k \in \mathbb{N}\)

When \(m \in \mathbb{R}\) we put \(0 \leq k < 0\)
Next identity

\[
\binom{x}{k} = \binom{x-1}{k} + \binom{x-1}{k-1}, \quad x \in \mathbb{R}, \quad k \in \mathbb{R}
\]

Case 1: Assume \( x \in \mathbb{N} \), i.e. \( m \in \mathbb{N} \)

\[
\binom{m}{k} = \binom{m-1}{k} + \binom{m-1}{k-1}
\]

And we have a combinatorial interpretation:

\[
\binom{m}{k} = \text{# of } k\text{-element subsets chosen from the } m\text{-element set}
\]

If we have \( m \)-eggs, with exactly one bad egg, we have

\[
\binom{m-1}{k} \text{ selections that have good eggs}
\]

\[
\binom{m-1}{k-1} \text{ of them contain the bad egg because they leave } k-1 \text{ of } m-1 \text{ good eggs.}
\]
\[ \binom{m}{k} = \binom{m-1}{k} + \binom{m-1}{k-1} \quad \text{for all } x \in \mathbb{N} \]

Proved

so

all eggs

only good

contain bad egg

\[ L(x) = \binom{x}{k} \] polynomial of the degree \( k \)

\[ P(x) = \binom{x-1}{k} + \binom{x-1}{k-1} \] polynomial of the degree \( k \)

We proved

\[ L(x) - P(x) = 0 \] for all \( x \in \mathbb{N} \)

by then 2

\[ L(x) - P(x) = 0 \] for all \( x \in \mathbb{R}, k+1 \)

\[ L(x) = P(x) \] for all \( x \in \mathbb{R}, k+1 \)
**Proof**

Use identities 3 + 2, we know:

3. \((x-k)(\frac{x}{k}) = x(\frac{x-1}{k-1})\), \(k(\frac{x}{k}) = x(\frac{x-1}{k-1})\)

**Add them**: 3 + 2

\[(x-k)(\frac{x}{k}) + k(\frac{x}{k}) = x(\frac{x-1}{k-1}) + x(\frac{x-1}{k-1})\]

\[\binom{x}{k}(x-k+k) = x\left(\binom{x-1}{k} + \binom{x-1}{k-1}\right)\]

We get:

\[\binom{x}{k} = x\left(\binom{x-1}{k} + \binom{x-1}{k-1}\right)\]

Check: \(x = 0\)
Check case $x = 0$

\[
\binom{0}{k} = \begin{pmatrix} -1 \end{pmatrix} + \begin{pmatrix} -1 \end{pmatrix}_{k-1}
\]

1. $k < 0$
   
   \[L = \mathbb{R}\]
   
   $0 = 0 + 0$

   \[
   \begin{pmatrix} -1 \end{pmatrix}^k = \frac{(-1)^k}{k!}
   \]

   \[
   \begin{pmatrix} -1 \end{pmatrix}_{k-1}^k = \frac{(-1)^k}{(k-1)!}
   \]

2. $k = 0$
   
   \[
   \begin{pmatrix} 0 \end{pmatrix} = \begin{pmatrix} 0 \end{pmatrix} + \begin{pmatrix} -1 \end{pmatrix}
   \]

   \[1 = 1 + 0\]
3. \[k > 0; \quad k \geq 1\]

\[
\binom{0}{k} = \binom{-1}{k} + \binom{-1}{k-1}
\]

\[
\binom{-1}{k} = \frac{(-1)^{k+1}}{k!}
\]

\[
\binom{-1}{k-1} = \frac{(-1)^{k-1}}{(k-1)!}
\]

\[
L = \binom{0}{k} = \frac{0^k}{k!} = 0
\]

Evaluate

\[
P = \binom{-1}{k} + \binom{-1}{k-1} = \frac{(-1)^{k+1}}{k!} + \frac{(-1)^{k-1}}{(k-1)!}
\]

\[
= \frac{(-1)^{k+1}(k-1) + (-1)^{k-1}k}{(k-1)!}
\]

\[
= \frac{(-1)^{k+1}(k-1) + (-1)^{k-1}k}{(k-1)!}
\]

\[
= \frac{(-1)^k}{k!}
\]

\[
= \frac{(-1)^{k-1}}{(k-1)!}
\]

\[
= (-1)^{k-1}(-k)
\]

\[
\frac{m+n}{m-n} = x^{m-n}(x-m)
\]

\[
\binom{k}{k-1} = \frac{k!}{(k-1)!}
\]

\[
= (-1)^{k-1}(k-1-k+1)
\]

\[
= (-1)^{k-1}(-k)
\]

\[
L = P
\]
Proof 3

\[ \binom{x-1}{k} + \binom{x-1}{k-1} = \frac{(x-1)^{k}}{k!} + \frac{(x-1)^{k-1}}{(k-1)!} \]

\[ = \frac{(x-1)^{k-1}}{k!} \cdot (x-k) + \frac{(x-1)^{k-1}}{k!} \cdot k \]

\[ = \frac{(x-1)^{k-1}}{k!} \cdot (x-k+k) \]

\[ = \frac{x \cdot (x-1)^{k-1}}{k!} = \frac{x}{k!} = \binom{x}{k} \]

\[ k \geq 0, \quad k-1 \geq 0, \quad k \geq 1 \]
\[
\begin{align*}
\binom{5}{3} &= \binom{4}{3} + \binom{4}{2} \\
&= \binom{4}{3} + \binom{3}{2} + \binom{3}{1} \\
&= \binom{4}{3} + \binom{3}{2} + \binom{2}{1} + \binom{2}{0} \\
&= \binom{4}{3} + \binom{3}{2} + \binom{2}{1} + \binom{1}{0} + \binom{0}{0} \\
&= \binom{4}{3} + \binom{3}{2} + \binom{2}{1} + \binom{1}{0} + \binom{0}{0} \\
&= \sum_{k=0}^{n} \binom{x+k}{k} = \binom{x}{0} + \binom{x+1}{1} + \ldots + \binom{x+n}{n} = \binom{x+n+1}{n+1} \\
0 \leq k \leq n & & \text{where } k < 0 \text{ factors are 0. Guess}
\end{align*}
\]
We guessed

\[ \sum_{k \leq n} \binom{x+k}{k} = \binom{x+n+1}{n} \quad \text{for } n \in \mathbb{Z}, \quad x \in \mathbb{R} \]

We use now formula

\[ \binom{x}{k} = \binom{x-1}{k} + \binom{x-1}{k-1} \]

to unfold (differently than before)

\[ \binom{5}{3} = \binom{4}{3} + \binom{4}{2} = \binom{3}{3} + \binom{3}{2} + \binom{4}{2} \]

\[ = \binom{2}{2} + \binom{3}{2} + \binom{4}{2} \]

\[ = \binom{1}{2} + \binom{2}{2} + \binom{3}{2} + \binom{4}{2} \]

\[ = \binom{0}{2} + \binom{1}{2} + \binom{2}{2} + \binom{3}{2} + \binom{4}{2} \]

\[ = \binom{0}{2} + \binom{0}{2} + \binom{2}{2} + \binom{3}{2} + \binom{4}{2} \]

And we guess a new formula:

\[ \sum_{k \leq n} \binom{k}{m} = \binom{n+1}{m+1} \]
We guess

\[ \sum_{0 \leq k \leq n} \binom{k}{m} = \binom{n+1}{m+1} \]

Both guesses 1 and 2 can be proved by math induction. We will prove 2 by induction and then we will prove 1 from 2.

**Combination Interpretation of 2:**

If we want to choose \( m+1 \) tickets from a set of \( m+1 \) tickets numbered 0, 1, ..., \( m \), there are \( \binom{m+1}{m} \) ways of doing this.

When the largest ticket selected

\[ \binom{0}{m} + \binom{1}{m} + \binom{2}{m} + \cdots + \binom{m}{m} = \binom{n+1}{m+1} \]
Proof by induction of $\binom{m}{k}$

\[ \sum_{k=0}^{m} \binom{k}{m} = \binom{m+1}{m+1} \]

Induction over $n \in \mathbb{N}$

\[ \begin{align*}
  n=0 & \quad \sum_{k=0}^{0} \binom{k}{m} = \binom{0}{m} = \binom{1}{m+1} = (0) = 1 \\
  \binom{1}{m+1} = \binom{0}{m+1} + \binom{0}{m} = 0, \quad \binom{0}{m} = 0 \quad m \geq 0 \\
\end{align*} \]

**Ind**

Assume

\[ S_{n-1} = \binom{n-1}{m+1} \]

Prove

\[ S_n = \sum_{k=0}^{n} \binom{k}{m} = \binom{n+1}{m+1} \]

Use:

\[ \binom{x}{k} = \binom{x-1}{k} + \binom{x-1}{k-1} \]

\[ \begin{align*}
  x &= n+1 \\
  k &= m+1 \\
  k &= 1 = m \\
\end{align*} \]

\[ \binom{n}{n+1} = \binom{m}{m+1} + (m) \]

\[ \begin{align*}
  n &= x-1 \\
  m &= x-1 \\
\end{align*} \]

\[ \binom{n}{m+1} \neq \binom{m}{m+1} + (m) \]

**Yes**

\[ m \geq 0 \quad \binom{1}{m} = 1 = \binom{0}{0} = 1 \]
Proof of 1 from 2:

1. \[
\sum_{k \leq n} \binom{m+k}{k} = \binom{m+n+1}{n}
\]
   \[n \in \mathbb{Z} \quad m, m+n \in \mathbb{N}\]

2. \[
\sum_{k=0}^{m} \binom{k}{m} = \binom{m+1}{m+1}
\]

Use \[
\binom{m}{k} = \binom{m}{n-k} \quad n \in \mathbb{N} \quad k \in \mathbb{Z}
\]

Evaluate for \[m+k \geq 0, \quad k \leq -m\] but terms are 0

Replace \(k \rightarrow k-m\):

\[-m \leq k \leq n\]

\[-m \leq k-m \leq n\]

\[0 \leq k \leq m+n\]

\[\sum_{k=m}^{m+n-1} \binom{m+n+1}{m+1} = \binom{m+n+1}{m+1} - \binom{m+n+1}{m-1} = m\]
Consider (2) \[ \sum_{k=0}^{m} \binom{k}{m} = \binom{m+1}{m+1} \]

PROVED \[ m, n \in \mathbb{N} \]

Consider \[ m = 1 \]

\[ \binom{m+1}{2} = \sum_{k=0}^{m} \binom{k}{1} = (0) + (1) + (2) + \ldots + (m) \]

\[ = 0 + 1 + 2 + \ldots + m = \frac{m(m+1)}{2} \]

General case \[ \binom{k}{m} \]:

\[ \sum_{k=0}^{m} \binom{k}{m} = \sum_{k=0}^{m} \frac{k}{m!} \]

\[ = \frac{1}{m!} \sum_{k=0}^{m} k \]

\[ = \sum_{k=0}^{m} k = \frac{m(m+1)}{2} \]

\[ = \binom{m+1}{m+1} \]

\[ \sum_{k=0}^{m} k^m = m! \left( \frac{m+1}{m+1} \right) \]

MAKE c.d.

FORMULA
\[ \sum_{k=0}^{m} k^m = m! \left( \frac{m+1}{(m+1)!} \right) \]

\[ = m! \left( \frac{m+1}{(m+1)!} \right) \]

\[ = \frac{(m+1)^{m+1}}{m+1} \]

**SPECIAL SUM.**

\[ \sum_{k=0}^{m} k^m = \frac{(m+1)^{m+1}}{(m+1)} \]

**n, m \in \mathbb{N}**

You can get it by integration (Simpson)

Look back at:

\[ \binom{x}{k} = \binom{x-1}{k} + \binom{x-1}{k-1} \]

\[ \binom{x+1}{m} = \binom{x}{m} + \binom{x}{m-1} \]

\[ \Delta \left( \binom{x}{m} \right) = \binom{x+1}{m} - \binom{x}{m} = \binom{x}{m-1} \]

**DEFINITION**

re-rewrite it as

\[ x = x + 1 \]

and evaluate

\[ \binom{x}{m} \]
\[ \Delta \binom{x}{m} = \binom{x}{m-1} \quad x \in \mathbb{R}, m \in \mathbb{Z} \]

and "integral"

\[ \sum \binom{x}{m} dx = \binom{x}{m+1} + C \]

We get a short proof of (2):

\[ \sum_{k=0}^{m} \binom{x}{k} \left[ \binom{x}{k+1} \right]_{0}^{m+1} = \binom{m+1}{k+1} - \binom{0}{k+1} = \binom{m+1}{k+1} \]

\[ \sum_{k=a}^{b} g(k) = \sum_{x=a}^{b} g(x) dx \]

used