Chapter 3

**INTEGER FUNCTIONS**

**Floor** For any \( x \in \mathbb{R} \) (real) we define:

\[
\lfloor x \rfloor = \text{the greatest integer less than or equal to } x
\]

**Ceiling**

\[
\lceil x \rceil = \text{the least integer greater than or equal to } x
\]

**Symbolic**

**Floor**

\[
\lfloor x \rfloor = \max \{ a \in \mathbb{Z} : a \leq x \}
\]

**Ceiling**

\[
\lceil x \rceil = \min \{ a \in \mathbb{Z} : a \geq x \}
\]

\( P \) has unique max \( \iff \) \( \exists ! a \in \mathbb{Z} : a \leq x \)

\( P \) has unique min \( \iff \) \( \exists ! a \in \mathbb{Z} : a \geq x \)
Fact: For any $x \in \mathbb{R}$, $\lfloor x \rfloor$, $\lceil x \rceil$ exist and are unique.

We can hence define functions $f_1 : \mathbb{R} \to \mathbb{Z}$ (floor) and $f_2 : \mathbb{R} \to \mathbb{Z}$ (ceiling):

- $f_1(x) = \lfloor x \rfloor = \max \{ a \in \mathbb{Z} : a \leq x \}$
- $f_2(x) = \lceil x \rceil = \min \{ a \in \mathbb{Z} : a \geq x \}$

Graph of $f_1$, $f_2$:

We read

$\lfloor 2 \rfloor = 2 \quad \lceil 3 \rceil = 3$
Properties of $\mathbb{L} \times \mathbb{L}$, $\mathbb{R} \times \mathbb{R}$

1. $[x] = x \text{ if } x \in \mathbb{Z}$
   $\mathbb{R} = x + 1 \text{ if } x \in \mathbb{Z}$

2. $x - 1 < x \leq x \leq x + 1$

3. $[x - 1] = [x - 1]$, $[x + 1] = [x + 1]$

4. $\mathbb{R} \times \mathbb{R} = \{ (0, x) : x \in \mathbb{Z} \}$

5. $\mathbb{L} \times \mathbb{L} = \{ (n, x) : n \leq x < n + 1 \}$

6. $\mathbb{L} \times \mathbb{L} = \{ (n, x) : x - 1 < n \leq x \}$

7. $\mathbb{R} \times \mathbb{R} = \{ (n, x) : n - 1 < x \leq n \}$

8. $\mathbb{R} \times \mathbb{R} = \{ (n, x) : x \leq n < x + 1 \}$

9. $\mathbb{L} \times \mathbb{R} = \mathbb{L} \times \mathbb{L} + n$

But $x \neq [x]$ when $n = 2$, $x = \frac{1}{2}$

$\lfloor \frac{1}{2} \rfloor = 0 + 2 \lfloor \frac{1}{2} \rfloor = 0$
**More Properties**

$x \in \mathbb{R}, m \in \mathbb{Z}$

\(10\) \( x < n \) \( \implies \) \( \lfloor x \rfloor < n \)

\(11\) \( m < x \) \( \implies \) \( m \leq \lfloor x \rfloor \)

\(12\) \( x \leq m \) \( \implies \) \( \lfloor x \rfloor \leq m \)

\(13\) \( m \leq x \) \( \implies \) \( m \leq \lfloor x \rfloor \)

**Proof of 10**

\(\rightarrow\) let \( x < n \), so \( \lfloor x \rfloor \leq x \)

\(\rightarrow\) let \( \lfloor x \rfloor < n \) by \(\circ\) \( x - 1 < \lfloor x \rfloor \) i.e. \( x < \lfloor x \rfloor + 1 \)

by \( \lfloor x \rfloor < n \) we get \( \lfloor x \rfloor + 1 \leq n \) so we get

\( x < \lfloor x \rfloor + 1 \leq n \) and \( x < n \).

**Factorial Part of** \( x \); \( \{ x \} \)

**Adjoint**

\[ \{ x \} = x - \lfloor x \rfloor \]

Write

\[ x = 3x + 1 \]

\[ \{ x \} = x - \lfloor x \rfloor \]

**Fact 1**

\[ x = m + \theta, m \in \mathbb{Z} \] and \( 0 \leq \theta < 1 \)

then \( m = \lfloor x \rfloor \) and \( \theta = \{ x \} \)

\(\circ\) \( \lfloor x \rfloor = n \) \( \implies \) \( n \leq x < n + 1 \)

we get

\[ x = \lfloor x \rfloor + \theta \implies \theta = \{ x \} \]
We proved \[ |x + m| = |x| + m, \quad m \in \mathbb{R}, \quad x \in \mathbb{R} \]

**Question**

What happens when we consider \( L \cdot x + y \), \( x, y \in \mathbb{R} \)?

Let's look.

\[ x = L \cdot x + \{ x \}, \quad y = L \cdot y + \{ y \} \]

\[ |x + y| = |L \cdot x + L \cdot y + \{ x \} + \{ y \}| \]

\[ = |L \cdot x + L \cdot y| + |\{ x \} + \{ y \}| \] \( \rhd \)

and \( 0 < \{ x \} + \{ y \} < 2 \) so we get

\[ |x + y| = \begin{cases} 
|L \cdot x + L \cdot y| & \text{when } 0 < \{ x \} + \{ y \} < 1 \\
|L \cdot x + L \cdot y| + 1 & \text{when } 1 \leq \{ x \} + \{ y \} < 2 
\end{cases} \]
Example

1. Find \( \lceil \log_2 35 \rceil \)

Observe
\[
2^5 < 35 \leq 2^6
\]

\[
\log_2 2^5 < \log_2 35 \leq \log_2 2^6
\]

\[
5 < \log_2 35 \leq 6
\]

We get

\[
\lceil \log_2 35 \rceil = 6
\]

2. Find \( \lceil \log_2 32 \rceil \)

\[
2^4 < 32 \leq 2^5
\]

\[
4 < \log_2 32 \leq 5
\]

By \( \circ \) we get

\[
\lceil \log_2 32 \rceil = 5
\]
EXAMPLE

FIND \( \lfloor \log_2 35 \rfloor \), \( \lfloor \log_2 32 \rfloor \)

Observe

\( 2^5 < 35 < 2^6 \)

\( 5 \leq \log_2 35 < 6 \)

\( \lfloor \log_2 35 \rfloor = 5 \)

\( \lfloor \log_2 32 \rfloor = 5 \)

\( \lfloor \log_2 32 \rfloor = \lfloor \log_2 327 \rfloor \)

Observe:

\( 35 \) has 6 digits in binary expansion and \( \lfloor \log_2 35 \rfloor = 6 \)

\( \lfloor \log_2 35 \rfloor = 5 \)

\( \lfloor \log_2 32 \rfloor = 5 \)

\( \lfloor \log_2 32 \rfloor = \lfloor \log_2 327 \rfloor \)

Question: Is \( \# \text{ digits} = \lfloor \log_2 n \rfloor \) true/false?

32 = (100000)

and \( \lfloor \log_2 32 \rfloor = 5 + 6 \)
**Question:**
Can we develop a connection (formula) between \( \lfloor \log_2 n \rfloor \) and the number of digits \( m \) of the binary representation of \( n \)? \((m \geq 0)\)

**Yes**
Let \( m+0, n \in \mathbb{N} \) such that \( n \) has \( m \) bits in binary representation. Hence we have

\[
2^{m-1} \leq n < 2^m
\]

\[
m-1 \leq \log_2 n < m
\]

and

\[
\lfloor \log_2 n \rfloor = m - 1
\]

Exercise

**DO THE SAME FOR**

\[
\lceil \log_2 n \rceil
\]

\( m = 32 \)

\( m = \lfloor \log_2 32 \rfloor + 1 = 5 + 1 = 6 \)  

\( m = \lfloor \log_2 546 \rfloor + 1 = 9 + 1 = 10 \)
Exercise

Prove that

\[ \forall (x \in \mathbb{R}, x \geq 0) \quad \lfloor \sqrt{x} \rfloor = \lfloor \sqrt{x} \rfloor \]

i.e.

\[ \forall x (x \in \mathbb{R}, x \geq 0 \implies \lfloor \sqrt{x} \rfloor = \lfloor \sqrt{x} \rfloor) \]

or just simply

\[ \lfloor \sqrt{x} \rfloor = \lfloor \sqrt{x} \rfloor \quad \text{for all} \quad x \in \mathbb{R}, \quad x \geq 0 \]

Fact²

\[ \lfloor \sqrt{x} \rfloor = \lfloor \sqrt{x} \rfloor \quad \text{for all} \quad x \in \mathbb{R}, \quad x \geq 0 \]

Proof:

Take \( \lfloor \sqrt{x} \rfloor \). First we get rid of outside \( \lfloor x \rfloor + \alpha \sqrt{x} \) and then of \( \lfloor x \rfloor \).

Let

\[ m = \lfloor \sqrt{x} \rfloor \]

\[ m \leq \sqrt{x} < m + 1 \]

\[ m^2 \leq x < (m+1)^2 \]

\[ m^2 \leq x < (m+1)^2 \]

\[ m = m \quad \text{and} \]

\[ \lfloor \sqrt{x} \rfloor = \lfloor \sqrt{x} \rfloor \]

ed.
we prove in a similar way (Exercise!)  

**FACT 3**  
\[ \sqrt[3]{x} = \sqrt[3]{x} \]  
for all \( x \in \mathbb{R}, x \geq 0 \)

**QUESTION:** \( \sqrt{x} \) is a particular \( f: \mathbb{R}^\times \rightarrow \mathbb{R} \)  
\( f(x) = \sqrt{x} \)  
(\text{can we have a similar property for other functions } \phi: \mathbb{R} \rightarrow \mathbb{R} \text{ (which?)})

**ANSWER:** YES. When \( f \) is monotonie and continuous and increasing! i.e. we will prove:  

**FACT 4**  
Let \( f: \mathbb{R}^\prime \rightarrow \mathbb{R} \) (maybe \( \mathbb{R}^\prime = \mathbb{R}, \mathbb{R}^\prime = \mathbb{R}^+ \text{ etc.} \)  
\( f \geq f(x) \) be such that \( f \) is continuous, monotonic and increasing on its domain \( \mathbb{R}^\prime \).

If additionally \( f \) has the following property  

\[ \text{If } f(x) \in \mathbb{Z}, \text{ then } x \in \mathbb{Z} \]  

then

\[ \lfloor f(x) \rfloor = \lfloor f(l \times r) \rfloor \text{ and} \]

\[ \lceil f(x) \rceil = \lceil f(l \times T) \rceil \]  
for all \( x \in \mathbb{R}^\prime \) for which \( \text{P} \) holds
Proof: \[ f(Γx) = Γf(x) Γ \]

Under the assumption:

1. Monotone and continuous
2. Trivial \( x = Γx \)

1. \( x = Γx \) we get

\[ f(x) = Γf(Γx) \]

This is trivial since \( x = 2 \) and

2. \( x = Γx \). By definition \( x < Γx \) and by monotonicity \( f(x) ≤ f(Γx) \), and by non-decreasing \( Γx \) ( \( x < y \) then \( Γx < Γy \) ) we get

\[ f(x) ≤ f(Γx) . \]

Now we show that \( < \) is impossible - hence we will have "=". Assume \( f(x) < f(Γx) \). By \( x = Γx \) we get

\[ f(x) < f(x) < f(Γx) . \]

If it continuous, then there is \( y \), such that

\[ f(y) = Γf(y) \]

and

\[ f(x) < f(y) < f(Γx) \]

so there holds when

\[ x < y < Γx \]

but \( x ≠ Γx \)

so we get

\[ x ≤ y < Γx \] (there is such a \( y \) !)

But \( f(y) = Γf(x) \), i.e. \( f(y) ∈ Z \), hence by \( y = 2 \) we get:

\[ y = 2 \] (there is no \( y ∈ 2 \).

These are contradictory! \( x ≤ y < Γx \) and
Special case of Fact 4 (for \( l \geq 1 \))

\[
\left\lfloor \frac{x + m}{n} \right\rfloor = \left\lfloor \frac{x}{n} \right\rfloor + m
\]

\[
\left\lceil \frac{x + m}{n} \right\rceil = \left\lceil \frac{x}{n} \right\rceil + m
\]

\[f(x) = \frac{x + m}{n}, \quad n, m \geq 2, \quad m > 0\]

Fact 5

\[f(x) = \frac{x}{m} + \frac{m}{n}\]

Example:

Take \( m = 0 \), \( n = 10 \)

Evaluate:

\[
\left\lfloor \frac{x}{10} \right\rfloor /10 = \left\lfloor \frac{x/10}{10} \right\rfloor = \left\lfloor \frac{x}{100} \right\rfloor = \left\lfloor \frac{x}{1000} \right\rfloor
\]

Dividing \( x \) three times by 10 and throwing off digits is the same as dividing \( x \) by 1000 and throwing out the remainder.
Integers in the INTERVALS

In interval (closed)

\[ [a, b] = \{ x \in \mathbb{R} : a \leq x \leq b \} \]

Standard notation = \([a .. b] \) - book notation

We use book notation, because

\([p(m)] \) denotes (in the book) the characteristic function

\[ (a, b) = \{ x \in \mathbb{R} : a < x < b \} = (a .. b) \]

Open interval

\[ (a, b) = \{ x \in \mathbb{R} : a \leq x < b \} = [a .. b) \]

Half-open interval

\[ (a, b] = \{ x \in \mathbb{R} : a < x \leq b \} = (a .. b] \]

Problem:

\[ A = \{ x \in \mathbb{Z} : 1 \leq x \leq 11 \} \]

Find: \[ |A| \]

How many are there integers in the intervals of real numbers
We bring back our $[\alpha, \beta]$ properties

\[
\begin{align*}
\alpha < n < \beta & \iff \lfloor \alpha \rfloor \leq n < \lfloor \beta \rfloor \\
\alpha < n \leq \beta & \iff \lfloor \alpha \rfloor < n \leq \lfloor \beta \rfloor
\end{align*}
\]

$[d \ldots \beta)$ contains exactly $[\lfloor \beta \rfloor - \lfloor d \rfloor + 1$ integers

$[d \ldots \beta]$ contains $[\lfloor \beta \rfloor - \lfloor d \rfloor ]+1$ int. AND we must assume $d \neq \beta$ to evaluate.

$[d \ldots \beta)$ contains $[\lfloor \beta \rfloor - \lfloor d \rfloor ] - 1$ (becomes $(d \ldots d) = \emptyset$ and can't contain $-1$ int)

\begin{array}{|c|c|c|c|}
\hline
[p\ldots p] & \lfloor p \rfloor - \lfloor d \rfloor + 1 & \alpha < \beta \\
[d\ldots p] & \lfloor p \rfloor - \lfloor d \rfloor & \alpha < \beta \\
[d\ldots p] & \lfloor p \rfloor - \lfloor d \rfloor & d < \beta \\
(d\ldots p) & [\lfloor p \rfloor - \lfloor d \rfloor ] - 1 & d < \beta \\
\hline
\end{array}
CASINO PROBLEM

There is a roulette wheel with 1,000 slots (numbered 1 \ldots 1,000).

If the number \( m \) that comes up on a spin is divisible by \( \left\lfloor \sqrt[3]{m} \right\rfloor \) i.e.

\[ \left\lfloor \sqrt[3]{m} \right\rfloor \mid m \]

then \( m \) is a winner.

In the game casino pays $5 if you are the winner; but the looser has to pay $1.

Can we expect to make money if we play this game?

Let's compute average winnings i.e. amount we win (or lose) per play.

\[ W - \# \text{ of winners} \]

\[ L = 1000 - W \# \text{ of losers} \]

If each number comes once during 1000 plays, we win 5W and lose L dollars.
**AVERAGE WINNINGS in 1000 plays**

\[
\frac{5W - L}{1000} = \frac{5W - (1000 - W)}{1000} = \frac{6W - 1000}{1000}
\]

We have advantage if

\[
\frac{6W - 1000}{1000} > 70, \quad 6W > 1000, \quad W > 166.7
\]

**Answer:** If there is 167 or more winners (and each number comes up only once) then we have the advantage, otherwise the casino wins.

**Problem:**

How to count the number of winners among 1 to 1000?

**Method:**

Use summation

\[
W = \sum_{n=1}^{1000} \{ n \text{ is a winner} \}
\]
The only "difficult" maneuver is the decision between lines (3) and (5) to treat \( m \leq 1000 \) as a special case.

(The max. \( k^3 \leq n \leq (k+1)^3 \) does not combine easily with \( 1 \leq n \leq 1000 \) when \( k = 10 \).)