CHAPTER 2
PART 5: INFINITE SUMS (SERIES)

Here are Definitions, Basic Theorems and Examples you must know
Series
Definitions, Theorems, Simple Examples

Must Know STATEMENTS- do not need to PROVE the Theorems

Definition
If the limit \( \lim_{n \to \infty} S_n \) exists and is finite, i.e.

\[
\lim_{n \to \infty} S_n = S,
\]

then we say that the infinite sum \( \sum_{n=1}^{\infty} a_n \) converges to \( S \) and we write

\[
\sum_{n=1}^{\infty} a_n = \lim_{n \to \infty} \sum_{k=1}^{n} a_k = S,
\]

otherwise the infinite sum diverges
Show

The infinite sum \( \sum_{n=1}^{\infty} (-1)^n \) diverges

The infinite sum \( \sum_{n=0}^{\infty} \frac{1}{(k+1)(k+2)} \) converges to 1
Definitions, Theorems, Simple Examples

**Theorem 1**
If the infinite sum

\[ \sum_{n=1}^{\infty} a_n \] converges, then \( \lim_{n \to \infty} a_n = 0 \)

**Definition 5**
An infinite sum

\[ \sum_{n=1}^{\infty} (-1)^{n+1} a_n, \text{ for } a_n \geq 0 \]

is called **alternating infinite sum** (alternating series)
Theorem 6  Comparing the series

Let \( \sum_{n=1}^{\infty} a_n \) be an infinite sum and \( \{b_n\} \) be a sequence such that

\[
0 \leq b_n \leq a_n \quad \text{for all } n
\]

If the infinite sum \( \sum_{n=1}^{\infty} a_n \) converges

then \( \sum_{n=1}^{\infty} b_n \) also converges and

\[
\sum_{n=1}^{\infty} b_n \leq \sum_{n=1}^{\infty} a_n
\]

Use Theorem 6 to prove that the series,

\[
\sum_{n=1}^{\infty} \frac{1}{(n+1)^2}
\]

converges
Definitions, Theorems, Simple Examples

Theorem 7  (D’Alambert’s Criterium)

If \( a_n \geq 0 \) and \( \lim_{n \to \infty} \frac{a_{n+1}}{a_n} < 1 \)

then the series \( \sum_{n=1}^{\infty} a_n \) converges

Theorem 8  (Cauchy’s Criterium)

If \( a_n \geq 0 \) and \( \lim_{n \to \infty} \sqrt[n]{a_n} < 1 \)

then the series \( \sum_{n=1}^{\infty} a_n \) converges
Definitions, Theorems, Simple Examples

**Theorem 9**  (Divergence Criteria)

If $a_n \geq 0$ and $\lim_{n \to \infty} \frac{a_{n+1}}{a_n} > 1$ or $\lim_{n \to \infty} \sqrt[n]{a_n} > 1$

then the series $\sum_{n=1}^{\infty} a_n$ diverges

Prove

The series $\sum_{n=1}^{\infty} \frac{1}{(n+1)^2}$ does not react on D’Alambert’s Criterium (Theorem 7)
Definitions, Theorems, Simple Examples

STUDY ALL EXAMPLES for Lecture 10
CHAPTER 3
INTEGER FUNCTIONS

Here is the **proofs** in course material you need to know for **Midterm 2** and **Final**

Plus the regular Homeworks Problems
PART1: Floors and Ceilings

Prove the following

Fact 3
For any \( x, y \in R \)

\[ [x + y] = [x] + [y] \quad \text{when} \quad 0 \leq \{x\} + \{y\} < 1 \]

and

\[ [x + y] = [x] + [y] + 1 \quad \text{when} \quad 1 \leq \{x\} + \{y\} < 2 \]

Fact 5
For any \( x \in R, \ x \geq 0 \) the following property holds

\[ \left\lfloor \sqrt{[x]} \right\rfloor = \left\lfloor \sqrt{x} \right\rfloor \]
PART1: Floors and Ceilings

Prove the following properties of characteristic functions

**F1** For any predicates $P(k), Q(k)$

$$[P(k) \cap Q(k)] = [P(k)][Q(k)]$$

**F2** For any predicates $P(k), Q(k)$

$$[P(k) \cup Q(k)] = [P(k)] + [Q(k)] - [P(k) \cap Q(k)]$$
PART1: Floors and Ceilings

Prove the Combined Domains Property

Property 4

\[ \sum_{Q(k) \cup R(k)} a_k = \sum_{Q(k)} a_k + \sum_{R(k)} a_k - \sum_{Q(k) \cap R(k)} a_k \]

where, as before,

\[ k \in K \quad \text{and} \quad K = K_1 \times K_2 \cdots \times K_i \quad \text{for} \quad 1 \leq i \leq n \]

and the above formula represents single \((i = 1)\) and multiple \((i > 1)\) sums
PART1: Floors and Ceilings

Study all 7 steps of our explanations to BOOK solution
I will give you **ONE to write in full** on the test

1. \[ W = \sum_{n=1}^{1000} [n \text{ is a winner}] = \sum_{n=1}^{1000} \lfloor \frac{3}{n} \rfloor \mid n \]

2. \[ W = \sum_{k,n} \lfloor k = \lfloor \frac{3}{n} \rfloor \rfloor [k|n] [1 \leq n \leq 1000] \]

3. \[ W = \sum_{k,n,m} \left[ k^3 \leq n < (k+1)^3 \right] [n = km] [1 \leq n \leq 1000] \]

4. \[ W = 1 + \sum_{k,m} \left[ k^3 \leq km < (k+1)^3 \right] [1 \leq k < 10] \]

5. \[ W = 1 + \sum_{k,m} \left[ m \in \left[ k^2 \ldots \frac{(k+1)^3}{k} \right] \right] [1 \leq k < 10] \]

6. \[ W = 1 + \sum_{1 \leq k < 10} \left( \left[ k^2 + 3k + 3 + \frac{1}{k} \right] - \left[ k^2 \right] \right) \]

7. \[ W = 1 + \sum_{1 \leq k < 10} (3k + 4) = 1 + \frac{7 + 31}{2} \cdot 9 = 172 \]
PART 2: Spectrum Partitions

Prove the following properties

P1 \[ |R(k)| = \sum_k [R(k)] \]

P2 \[ \sum_k [R(k)] = \sum_{R(k)} 1 = |R(k)| \]

Justify that

\[ N(\alpha, n) = \sum_{k>0} \left[ k < \frac{n+1}{\alpha} \right] \]

Write a detailed proof of

\[ N(\alpha, n) = \left\lfloor \frac{n+1}{\alpha} \right\rfloor - 1 \]

Write a detailed proof of

Finite Fact

\[ |A_n| + |B_n| = n \quad \text{for any } n \in N - \{0\} \]
PART2: Spectrum Partitions

Prove the following

Fact P2
If $|A| + |B| = |X|$ and $A \neq \emptyset$, $B \neq \emptyset$ and $A \cap B = \emptyset$
then the sets $A, B$ form a finite partition of $X$

Spectrum Fact

$$Spec(\sqrt{2}) \cap Spec(2 + \sqrt{2}) = \emptyset$$

Finite Spectrum Partition Theorem
1. $A_n \neq \emptyset$ and $B_n \neq \emptyset$
2. $A_n \cap B_n = \emptyset$
3. $A_n \cup B_n = \{1, 2, \ldots n\}$
PART2: Spectrum Partitions

Prove - use your favorite proof out of the two I have provided

Spectrum Partition Theorem
1. \( \text{Spec}(\sqrt{2}) \neq \emptyset \) and \( \text{Spec}(2 + \sqrt{2}) \neq \emptyset \)
2. \( \text{Spec}(\sqrt{2}) \cap \text{Spec}(2 + \sqrt{2}) = \emptyset \)
3. \( \text{Spec}(\sqrt{2}) \cup \text{Spec}(2 + \sqrt{2}) = \mathbb{N} - \{0\} \)
PART3: Sums

Write detailed evaluation of

$$\sum_{0 \leq k < n} \lfloor \sqrt{k} \rfloor$$

Hint: use

$$\sum_{0 \leq k < n} \lfloor \sqrt{k} \rfloor = \sum_{0 \leq k < n} \sum_{m \geq 0, \ m = \lfloor \sqrt{k} \rfloor} m$$
Chapter 4 Material in the Lecture 12
Theorems, Proofs and Problems

**JUSTIFY** correctness of the following example and be ready to do similar problems upwards or downwards

Represent $19151$ in a system with base $12$

**Example**

$$19151 = 1595 \cdot 12 + 11$$

$$1595 = 132 \cdot 12 + 11$$

$$132 = 11 \cdot 12 + 0$$

$a_0 = 11, \quad a_1 = 11, \quad a_2 = 0, \quad a_3 = 11$

So we get

$$19151 = (11,0,11,11)_{12}$$
Write a proof of Step 1 or Step 2 of the Proof of the Correctness of Euclid Algorithm

You can use Lecture OR BOOK formalization and proofs

Use Euclid Algorithms to prove

When a product ac of two natural numbers is divisible by a number b that is relatively prime to a, the factor c must be divisible by b

Use Euclid Algorithms to prove

\[ \gcd(ka, kb) = k \cdot \gcd(a, b) \]
Theorems, Proofs and Problems

Prove:
Any common multiple of $a$ and $b$ is divisible by $\text{lcm}(a,b)$

Prove the following

$$\forall p, q_1, q_2, \ldots q_n \in P \ (p \mid \prod_{k=1}^{n} q_k \Rightarrow \exists_{1 \leq i \leq n} (p = q_i))$$

Write down a formal formulation (in all details) of the Main Factorization Theorem and its General Form
Theorems, Proofs and Problems

Prove that the representation given by Main Factorization Theorem is unique.

Explain why and show that $18 = \langle 1, 2 \rangle$.

Prove

\[ k = \gcd(m, n) \quad \text{if and only if} \quad k_p = \min\{m_p, n_p\} \]

\[ k = \text{lcm}(m, n) \quad \text{if and only if} \quad k_p = \max\{m_p, n_p\} \]

Let

\[ m = 2^0 \cdot 3^3 \cdot 5^2 \cdot 7^0 \quad n = 2^0 \cdot 3^1 \cdot 5^0 \cdot 7^3 \]

Evaluate $\gcd(m, n)$ and $k = \text{lcm}(m, n)$.
Exercises

1. Use Facts 6-8 to prove Theorem 5
For any $a, b \in \mathbb{Z}^+$ such that $\text{lcm}(a, b)$ and $\text{gcd}(a, b)$ exist

$$\text{lcm}(a, b) \cdot \text{gcd}(a, b) = ab$$

2. Use Theorem 5 and the BOOK version of Euclid Algorithm to express $\text{lcm}(n \mod m, m)$ when $n \mod m \neq 0$
This is Ch4 Problem 2