PART 1: ORDER RELATIONS

Order Relation \( R \subset A \times A \) is an order on \( A \) iff \( R \) is 1. Reflexive, 2. Antisymmetric, 3. Transitive, i.e.

1. \( \forall a \in A \) \( (a, a) \in R \)
2. \( \forall a, b \in A \) \( ((a, b) \in R \cap (b, a) \in R \Rightarrow a = b) \)
3. \( \forall a, b, c \in A \) \( ((a, b) \in R \cap (b, c) \in R \Rightarrow (a, c) \in R) \)

Total Order \( R \subset (A \times A) \) is a total order on \( A \) iff \( R \) is an order and any two elements of \( A \) are comparable, i.e. \( \forall a, b \in A \) \( ((a, b) \in R \cup (b, a) \in R) \).

Historical names Order is also called partial order and total order is also called a linear order.

Notations Order relations are usually denoted by \( \leq \). We use, in our lecture notes the notation \( \leq \). Our book, and hence this handout uses the notation \( \preceq \) as a symbol for order relation. Remember, that even if we use \( \leq \) as the order relation symbol, it is a SYMBOL for ANY order relation and not only a symbol for a natural order \( \leq \) in number sets.

Poset A set \( A \neq \emptyset \) ordered by a relation \( R \) is called a poset. We write it as a tuple: \( (A, R) \), \( (A, \leq) \) or \( (A, \preceq) \). Name poset stands for ”partially ordered set”.

Diagram Diagram or Hasse Diagram of order relation is a graphical representaion of a poset. (see lecture notes or the book for definition).

Special elements in a poset \( (A, \leq) \) (book notation) are: maximal, minimal, greatest (largest) and smallest (least) and are defined below.

Smallest (least) \( a_0 \in A \) is a smallest (least) element in the poset \( (A, \leq) \) iff \( \forall a \in A \) \( (a_0 \preceq a) \).

Greatest (largest) \( a_0 \in A \) is a greatest (largest) element in the poset \( (A, \leq) \) iff \( \forall a \in A \) \( (a \preceq a_0) \).

Maximal (formal) \( a_0 \in A \) is a maximal element in the poset \( (A, \leq) \) iff \( \neg \exists a \in A \) \( (a \preceq a_0 \land a \neq a_0) \).

Maximal (informal) \( a_0 \in A \) is a maximal element in the poset \( (A, \leq) \) iff on the diagram of \( (A, \leq) \) there is no element placed above \( a_0 \).

Minimal \( a_0 \in A \) is a minimal element in the poset \( (A, \preceq) \) iff \( \neg \exists a \in A \) \( (a \preceq a_0 \land a \neq a_0) \).

Minimal (informal) \( a_0 \in A \) is a minimal element in the poset \( (A, \preceq) \) iff on the diagram of \( (A, \preceq) \) there is no element placed below \( a_0 \).

Lower Bound Let \( B \subseteq A \) and \( (A, \preceq) \) is a poset. \( a_0 \in A \) is a lower bound of a set \( B \) iff \( \forall b \in B \) \( (a_0 \preceq b) \).

Upper Bound Let \( B \subseteq A \) and \( (A, \preceq) \) is a poset. \( a_0 \in A \) is an upper bound of a set \( B \) iff \( \forall b \in B \) \( (b \preceq a_0) \).

Least upper bound of \( B \) (lub \( B \)) Given: a set \( B \subseteq A \) of a poset \( (A, \preceq) \).
An element \( x_0 \in A \) is a least upper bound of \( B \), \( x_0 = \text{lub} B \) iff \( x_0 \) is (if exists) the least (smallest) element in the set of all upper bounds of \( B \), ordered by the poset order \( \preceq \).
Greatest lower bound of B (glb B)  Given: a set \( B \subseteq A \) a poset \((A, \leq)\).

An element \( x_0 \in A \) is a greatest lower bound of \( B \), \( x_0 = glbB \) iff \( x_0 \) is (if exists) the greatest element in the set of all lower bounds of \( B \), ordered by the poset order \( \leq \).

PART 2: LATTICES and BOOLEAN ALGEBRAS

Lattice  A poset \((A, \leq)\) is a lattice iff For all \( a, b \in A \) both \( lub\{a, b\} \) and \( glb\{a, b\} \) exist.

Lattice notation  Observe that by definition elements \( lubB \) and \( glbB \) are always unique (if they exist). For \( B = \{a, b\} \) we denote:
\[
lub\{a, b\} = a \cup b \quad \text{and} \quad glb\{a, b\} = a \cap b.
\]

Lattice union (meet)  The element \( lub\{a, b\} = a \cup b \) is called a lattice union (meet) of \( a \) and \( b \).
By lattice definition for any \( a, b \in A \) \( a \cup b \) always exists.

Lattice intersection (joint)  The element \( glb\{a, b\} = a \cap b \) is called a lattice intersection (joint) of \( a \) and \( b \).
By lattice definition for any \( a, b \in A \) \( a \cap b \) always exists.

Lattice as an Algebra  An algebra \((A, \cup, \cap)\), where \( \cup \), \( \cap \) are two argument operations on \( A \) is called a lattice iff the following conditions hold for any \( a, b, c \in A \) (they are called lattice AXIOMS):

1. \( a \cup b = b \cup a \) and \( a \cap b = b \cap a \)
2. \( (a \cup b) \cap c = a \cap (b \cup c) \) and \( (a \cap b) \cap c = a \cap (b \cap c) \)
3. \( a \cap (a \cup b) = a \) and \( a \cup (a \cap b) = a \).

Lattice axioms  The conditions 1-3 from above definition are called lattice axioms.

Lattice orderings  Let the \((A, \cup, \cap)\) be a lattice. The relations:
\[ a \leq b \iff a \cup b = b \quad \text{and} \quad a \leq b \iff a \cap b = a \]
are order relations in \( A \) and are called a lattice orderings.

Distributive lattice  A lattice \((A, \cup, \cap)\) is called a distributive lattice iff for all \( a, b, c \in A \) the following conditions hold

4. \( a \cup (b \cap c) = (a \cup b) \cap (a \cup c) \)
5. \( a \cap (b \cup c) = (a \cap b) \cup (a \cap c) \).

Distributive lattice axioms  Conditions 1-5 from above are called a distributive lattice axioms.

Lattice special elements  The greatest element in a lattice (if exists) is denoted by \( 1 \) and is called a lattice UNIT. The least (smallest) element in \( A \) (if exists) is denoted by \( 0 \) and called a lattice zero.

Lattice with unit and zero  If \( 0 \) (lattice zero) and \( 1 \) (lattice unit) exist in a lattice, we will write the lattice as: \((A, \cup, \cap, 0, 1)\) and call it a lattice with zero and unit.

Lattice Unit Definition  Let \((A, \cup, \cap)\) be a lattice. An element \( x \in A \) is called a lattice unit iff for any \( a \in A \) \( x \cap a = a \) and \( x \cup a = x \).

Lattice Unit Axioms  If such element \( x \) exists we denote it by \( 1 \) and we write the unit axioms as follows.
Let \( A, \cup, \cap \) be a lattice. An element \( x \in A \) is called a lattice zero iff for any \( a \in A \) \( x \cap a = x \) and \( x \cup a = a \).

We denote the lattice zero by 0 and write the zero axioms as follows.

\[
\begin{align*}
0 \cup a &= 0 \\
0 \cap a &= a.
\end{align*}
\]

Complement Definition Let \( A, \cup, \cap, 1, 0 \) be a lattice with unit and zero. An element \( x \in A \) is called a complement of an element \( a \in A \) iff \( a \cup x = 1 \) and \( a \cap x = 0 \).

Complement axioms Let \( A, \cup, \cap, 1, 0 \) be a lattice with unit and zero. The complement of \( a \in A \) is usually denoted by \( -a \) and the above conditions that define the complement above are called complement axioms. The complement axioms are usually written as follows.

\[
\begin{align*}
c1 & \quad a \cup -a = 1 \\
c2 & \quad a \cap -a = 0.
\end{align*}
\]

Boolean Algebra Definition A distributive lattice with zero and unit such that each element has a complement is called a Boolean Algebra.

Boolean Algebra Axioms A lattice \( A, \cup, \cap, 1, 0 \) is called a Boolean Algebra iff the operations \( \cap, \cup \) satisfy axioms \( l1-l5 \), \( 0 \in A \) and \( 1 \in A \) satisfy axioms \( l6-l9 \) and each element \( a \in A \) has a complement \( -a \in A \), i.e.

\[
l0 \quad \forall a \in A \exists -a \in A ((a \cup -a = 1) \cap (a \cap -a = 0)).
\]

SOME BASIC FACTS

Uniqueness In any poset \( (A, \preceq) \), if a greatest and a least elements exist, then they are unique.

Finite Posets If \( (A, \preceq) \) is a finite poset (i.e. \( A \) is a finite set), then a unique maximal (if exists) is the largest element and a unique minimal (if exists) is the least element.

Infinite Posets It is possible to to order an infinite set \( A \) in such a way that the poset \( (A, \preceq) \) has a unique maximal element (minimal element) and no largest element (least element).

Any poset In any poset, the largest element is a unique maximal element and the least element is the unique minimal element.

Lower, upper bounds A set \( B \subseteq A \) of a poset \( (A, \preceq) \) can have none, finite or infinite number of lower or upper bounds, depending of ordering.

Finite lattice If \( (A, \cup, \cap) \) is a finite lattice (i.e. \( A \) is a finite set), then 1 and 0 always exist.

Infinite lattice If \( (A, \cup, \cap) \) is an infinite lattice (i.e. the set \( A \) is infinite ), then 1 or 0 might or might not exist.

For example:

- \( (N, \preceq) \) is a lattice with 0 (the number 0) and no 1.
- \( (Z, \preceq) \) is a lattice without 0 and without 1.

Finite Boolean Algebra Non- generate Finite Boolean Algebras always have \( 2^n \) elements \( (n \geq 1) \).