Chapter 5, problem 8

cse547

Problem 8

- Evaluate

$$\sum_k \binom{n}{k} (-1)^k \left(1 - \frac{k}{n}\right)^n$$

What is the approximate value of this sum, when n is very large?

Formula 1

Textbook P.188 (5.40)

$$\Delta^n f(x) = \sum_k \binom{n}{k} (-1)^{n-k} f(x+k)$$

(integer $n \geq 0$)

Formula 2

Textbook P.189 (bottom)

If $f(x) = a_d x^d + a_{d-1} x^{d-1} + \ldots + a_1 x^1 + a_0 x^0$,

then

$$\Delta^n f(0) = \begin{cases} n! a_n & \text{if } n \le d \\ 0 & \text{if } n > d \end{cases}$$

Observation

Formula 1

$$\Delta^n f(x) = \sum_k \binom{n}{k} (-1)^{n-k} f(x+k)$$

Our problem

$$\sum_k \binom{n}{k} (-1)^k \left(1-\frac{k}{n}\right)^n$$

Evaluation (1/5)

Let
$$f(x) = (1 - \frac{x}{n})^n$$

Use Formula 1

$$\Delta^n f(x) = \sum_k \binom{n}{k} (-1)^{n-k} f(x+k)$$

$$\Delta^n f(x) = \sum_k \binom{n}{k} (-1)^{n-k} (1 - \frac{x+k}{n})^n$$

$$\Delta^n f(0) = \sum_k \binom{n}{k} (-1)^{n-k} (1 - \frac{k}{n})^n$$

Evaluation (2/5)

$$\Delta^n f(0) = \sum_k \binom{n}{k} (-1)^{n-k} (1 - \frac{k}{n})^n$$

$$= \sum_k \binom{n}{k} (-1)^{k-n} (1 - \frac{k}{n})^n$$

$$= (-1)^{-n} \sum_k \binom{n}{k} (-1)^k (1 - \frac{k}{n})^n$$

$$\sum_k \binom{n}{k} (-1)^k (1 - \frac{k}{n})^n = (-1)^n \Delta^n f(0)$$

Evaluation (3/5)

$$\sum_k \binom{n}{k}(-1)^k(1-\frac{k}{n})^n = (-1)^n \Delta^n f(0)$$

Use Formula 2

$$\Delta^n f(0) = n! a_n$$

$$f(x) = (1-\frac{x}{n})^n$$

(Binomial Expansion)

$$= 1 + (-1)^1 \binom{n}{1}(\frac{x}{n})^1 + (-1)^2 \binom{n}{2}(\frac{x}{n})^2 + \ldots + (-1)^n \binom{n}{n}(\frac{x}{n})^n$$

$$a_n = (-1)^n \binom{n}{n}(\frac{1}{n})^n = (-1)^n(\frac{1}{n})^n$$

Evaluation (4/5)

$$\Delta^n f(0) = n! a_n = \boxed{n!(-1)^n (\frac{1}{n})^n}$$

$$\sum_k \binom{n}{k} (-1)^k (1 - \frac{k}{n})^n = (-1)^n \Delta^n f(0)$$

$$= (-1)^n (-1)^n n! (\frac{1}{n})^n$$

$$= (-1)^{2n} \frac{n!}{n^n}$$

Evaluation (5/5)

When n is very large,

$$\sum_{k} \binom{n}{k} (-1)^k \left(1 - \frac{k}{n}\right)^n = 0$$

since

$$\lim_{n\to\infty} \frac{n!}{n^n} = \lim_{n\to\infty} \left\{ \frac{n}{n} \cdot \frac{(n-1)}{n} \cdot \frac{(n-2)}{n} \cdot \ldots \cdot \frac{1}{n} \right\} = 0$$