Chapter 5, Problem 7
Problem

- Is it true also when $k < 0$?

\[
r^k \left( r - \frac{1}{2} \right)^k = \frac{(2r)^{2k}}{2^{2k}}
\]
Observation 1 \((k > 0)\)

- Each term in the denominator of expanded \(r\) to the \(-k\) falling adds \(2r\) with an even number, increasingly.

\[
r^{-k} = \frac{1}{(r + 1)(r + 2)\ldots(r + k)}
= \frac{1}{\left(\frac{2r + 2}{2}\right)\left(\frac{2r + 4}{2}\right)\ldots\left(\frac{2r + 2k}{2}\right)}
= \frac{2^k}{(2r + 2)(2r + 4)\ldots(2r + 2k)}
\]
Observation 2 \( (k > 0) \)

- Each term in the denominator of expanded \((r-1/2)^k\) to the \(-k\) falling adds with an odd number, increasingly.

\[
(r - \frac{1}{2})^{-k} = \frac{1}{(r - \frac{1}{2} + 1)(r - \frac{1}{2} + 2) \cdots (r - \frac{1}{2} + k)}
\]

\[
= \frac{1}{\left( \frac{2r - 1 + 2}{2} \right) \left( \frac{2r - 1 + 4}{2} \right) \cdots \left( \frac{2r - 1 + 2k}{2} \right)}
\]

\[
= \frac{1}{\left( \frac{2r + 1}{2} \right) \left( \frac{2r + 3}{2} \right) \cdots \left( \frac{2r + 2k - 1}{2} \right)}
\]

\[
= \frac{2^k}{(2r + 1)(2r + 3) \cdots (2r + 2k - 1)}
\]
\[
\begin{align*}
\frac{2^k}{(2r + 2)(2r + 4)\ldots(2r + 2k)} & \times \\
\frac{2^k}{(2r + 1)(2r + 3)\ldots(2r + 2k - 1)} & = \\
\frac{2^{2k}}{(2r + 1)(2r + 2)(2r + 3)\ldots(2r + 2k - 1)(2r + 2k)}
\end{align*}
\]
This result equals to $2r$ to the $-2k$ falling times 

$$2^{2k}.$$ 

\[
\frac{2^{2k}}{(2r + 1)(2r + 2)(2r + 3)\ldots(2r + 2k - 1)(2r + 2k)}
\]
Thus,
\[ r^{-k} (r - \frac{1}{2})^{-k} = (2r)^{-2k} 2^{2k} = \frac{(2r)^{-2k}}{2^{-2k}}, k > 0 \]

As the problem asks for in case \( k < 0 \), we can set a \( k' \) whose domain is negative integers; therefore we can replace \(-k\) with \(k'\).

We can rewrite the formula as
\[ r^{k'} (r - \frac{1}{2})^{k'} = \frac{(2r)^{2k'}}{2^{2k'}}, k' < 0 \]
The Result

As the domain of $k'$ is as same as the domain of $k$ (that is less than zero) in the problem, we got the solution:

$$r^k(r - \frac{1}{2})^k = \frac{(2r)^{2k}}{2^{2k}}$$

is also true when $k < 0$. 
Verifying the property

- In case $k = -1$,

$$r^{-1}(r - \frac{1}{2})^{-1} = \frac{4}{(2r + 1)(2r + 2)} = \frac{(2)^{-2}}{2^{-2}}$$

- In case $k = -2$,

$$r^{-2}(r - \frac{1}{2})^{-2} = \frac{16}{(2r + 1)(2r + 2)(2r + 3)(2r + 4)} = \frac{(2)^{-4}}{2^{-4}}$$