CHAPTER 5
EXERCISE # 4

THE PROBLEM

EVALUATE:

$\binom{-1}{k}$ by negating its upper index.

(k **in** $\mathbf{Z}$)

What is Upper Index Negation ?

The following identity:

$$\binom{r}{k} = (-1)^k \binom{k-r-1}{k}$$

(where k **in** $\mathbf{Z}$ and r **in** $\mathbf{R}$)

is called the upper index negation identity.

(Proof of this identity is given in lecture notes-
slide 40 of chapter 5; and pg 164 of the book.)

SOLUTION TO THE PROBLEM

$$\binom{-1}{k} = (-1)^k \binom{k-(-1)-1}{k}$$

We are using the "Upper index negation Identity" here.

$$= (-1)^k \binom{k+1-1}{k}$$

$$= (-1)^k \binom{k}{k} \quad \ldots eq(1)$$

LET US EVALUATE THIS

EVALUATING SUB-PROBLEM

In order to be able to solve $\binom{k}{k}$, we need to first look at the following formal definition, “def1”:

$$\binom{r}{z} = 0 \quad \text{, if } z<0.$$

$$\binom{r}{z} = \frac{r^{\underline{k}}}{k!} \quad \text{, if } z \geq 0.$$

Z in **Z**

r in **R**

SOLUTION TO THE SUB-PROBLEM

Hence in order to be able to solve, $\begin{pmatrix} k \\ k \end{pmatrix}$,

(where k ranges over all integers), it is obvious from the definition above that we need to consider two cases. One case when $k < 0$, and another case when $k \geq 0$.

SOLUTION TO THE SUB-PROBLEM

- Case When k <0, (then by def 1):

$$\binom{k}{k} = 0$$

- Case When k ≥ 0, (then by def 1):

$$\binom{k}{k} = \frac{k^{\underline{k}}}{k!} = \frac{k(k-1)(k-2)\ldots(k-k+1)}{k!}$$

$$= \frac{k!}{k!} = 1$$

GOING BACK TO THE PROBLEM

- Now, we had proven before that:

$$\binom{-1}{k} = (-1)^k \binom{k}{k}$$

And we have just proven that:

$$\binom{k}{k} = 0 \quad \text{if } k < 0.$$

$$\binom{k}{k} = 1 \quad \text{if } k \geq 0.$$

SOLUTION TO THE PROBLEM

- Solution When k<0:

$$\binom{-1}{k} = (-1)^k \binom{k}{k} = (-1)^k (0) = \mathbf{0}$$

- Solution When k ≥ 0:

$$\binom{-1}{k} = (-1)^k \binom{k}{k}$$

$$= (-1)^k (1) = (-1)^k$$

FINAL ANSWER

- When k<0:

$$\binom{-1}{k} = 0$$

- When k ≥ 0:

$$\binom{-1}{k} = (-1)^k$$