Chapter 5, Problem 35

Evaluate the sum

\[ S_n = \sum_{k \leq n} \binom{n}{k} 2^{k-n} \]
Q. Evaluate the sum \( S_n = \sum_{k \leq n} \binom{n}{k} 2^{k-n} \)

Let \( S_n = \sum_{k \leq n} \binom{n}{k} 2^{k-n} \) \( \ldots \) \( \ldots \)(1)

Consider the binomial theorem,

\[(x + y)^r = \sum_{k \leq r} \binom{r}{k} x^k y^{r-k} \] \( \ldots \) \( \ldots \)(2)

Compare (1) and (2).
We will try to get the RHS of (1) to look like the RHS of (2)

\[ S_n = \sum_{k \leq n} \binom{n}{k} 2^{k-n} \]

or,

\[ S_n = \frac{1}{2^n} \sum_{k \leq r} \binom{r}{k} 2^k \]

or,

\[ S_n = \frac{1}{2^n} \sum_{k \leq r} \binom{r}{k} 2^k 1^{r-k} \]
\[ S_n = \frac{1}{2^n} \sum_{k \leq r} \binom{r}{k} 2^k 1^{r-k} \]

\[ (x + y)^r = \sum_{k \leq r} \binom{r}{k} x^k y^{r-k} \]

\[ S_n = \frac{1}{2^n} (1+2)^n = \frac{3^n}{2^n} \]