Chapter 5-Problem 18, 45

Chapter 15, problem 18

Problem description:

Evaluate $\binom{r}{k}\binom{r-1/3}{k}\binom{r-2/3}{k}$

$r \in N$

$k \in N$

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- From the textbook, we can see

$$\binom{r}{k}\binom{r-1/2}{k} = \binom{2r}{2k}\binom{2k}{k}\Big/2^{2k} \qquad (5.35)$$

- Lets review this

$$\binom{r}{k} = \begin{cases} \dfrac{r(r-1)\ldots(r-k+1)}{k(k-1)\ldots(1)} = \dfrac{r^{\underline{k}}}{k!} & ,k \geq 0 \\ 0 & ,k < 0 \end{cases}$$

where, r is real, and k are natural

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So, now let's go to the question

$$\binom{r}{k}\binom{r-1/3}{k}\binom{r-2/3}{k}$$

$$=\frac{r(r-1)...(r-k+1)}{k!}\frac{(r-1/3)...(r-1/3-k+1)}{k!}\frac{(r-2/3)...(r-2/3-k+1)}{k!}$$

$$=\frac{r(r-1)...(r-k+1)}{k!}\frac{(r-{}^1\!/\!_3)...(r-k+{}^2\!/\!_3)}{k!}\frac{(r-{}^2\!/\!_3)...(r-k+{}^1\!/\!_3)}{k!}$$

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$$= \frac{r(r-\frac{1}{3})(r-\frac{2}{3})(r-1)...(r-k+1)(r-k+\frac{2}{3})(r-k+\frac{1}{3})}{(k!)^3}$$

Multiply by 3^{3k}

$$= \frac{3r(3r-1)(3r-2)(3r-3)...(3r-3k+3)(3r-3k+2)(3r-3k+1)}{(k!)^3} \times \frac{1}{3^{3k}}$$

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$$= 3r(3r-1)(3r-2)(3r-3)...(3r-3k+1) \times \frac{1}{(k!)^3 \times 3^{3k}}$$

$$= \frac{(3r)!}{(3r-3k)!} \times \frac{1}{(k!)^3 \times (3)^{3k}}$$

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Now some thing is needed, as

$$\binom{3r}{3k}\binom{3k}{??}\cdots \Big/ 3^{3k}$$

$$= \frac{(3r)!}{(3r-3k)!} \times \frac{1}{(k!)^3 \times 3^{3k}}$$

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$$= \frac{(3r)!}{(3r-3k)!(3k)!} \times \frac{(3k)!}{(2k)!} \times \frac{(2k)!}{1} \times \frac{1}{(k!)^3 \times 3^{3k}}$$

$$= \frac{(3r)!}{(3r-3k)!(3k)!} \times \frac{(3k)!}{(2k)!k!} \times \frac{(2k)!}{(k!)^2} \times \frac{1}{3^{3k}}$$

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$$= \binom{3r}{3k}\binom{3k}{2k}\binom{2k}{k} \Big/ 3^{3k}$$

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Finally, the solution for the problem is

$$\binom{r}{k}\binom{r-\frac{1}{3}}{k}\binom{r-\frac{2}{3}}{k}=\binom{3r}{3k}\binom{3k}{2k}\binom{2k}{k}\Big/3^{3k}$$

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- Find a closed form for $\sum_{k \le n} \binom{2k}{k} 4^{-k}$

- Solution:

  Above summation can be rewritten as follows:

$$\sum_{k \le n} \binom{2k}{k} 4^{-k} = \sum_{k \le n} \binom{2k}{k} / 4^{k} = \sum_{k \le n} \binom{2k}{k} / 2^{2k}$$

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- Using identity (5.36) from textbook :

$$\binom{k-\frac{1}{2}}{k} = \binom{2k}{k} \Big/ 2^{2k} \quad \text{,integer } k$$

- We rearrange the problem as follows: $\sum_{k \le n} \binom{2k}{k} 4^{-k} = \sum_{k \le n} \binom{2k}{k} / 2^{2k} = \sum_{k \le n} \binom{k-\frac{1}{2}}{k}$

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- Using identity (5.9) from textbook:

$$\sum_{k\le n}\binom{r+k}{k}=\binom{r+n+1}{n} \quad ,\text{integer } n$$

- Putting r = -1/2 & k = k in above equation $\sum_{k\le n}\binom{k-1/2}{k}=\binom{-1/2+n+1}{n}=\binom{(n+1)-1/2}{n}$

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- Using identity (5.35) from textbook:

$$\binom{r}{k}\binom{r-\frac{1}{2}}{k} = \binom{2r}{2k}\binom{2k}{k}\Big/2^{2k}, \text{integer } k$$

- Putting r = n+1 & k = n in above equation

$$\binom{n+1}{n}\binom{(n+1)-\frac{1}{2}}{n} = \binom{2n+2}{2n}\binom{2n}{n}\Big/2^{2n}$$

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- Dividing by $\binom{n+1}{n}$ on both the sides we get,
- Also, $\binom{(n+1)-½}{n} = \dfrac{\binom{2n+2}{2n}\binom{2n}{n}}{\binom{n+1}{n} \bullet 2^{2n}}$

and $\binom{2n+2}{2n} = \dfrac{(2n+2)(2n+1)}{2} = (n+1)(2n+1)$

$\binom{n+1}{n} = n+1$

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- Solving further we have,

$$\frac{\binom{2n+2}{2n}\binom{2n}{n}}{\binom{n+1}{n}\bullet 2^{2n}} = \frac{(n+1)(2n+1)\binom{2n}{n}}{(n+1)\cdot 2^{2n}}$$

$$= \frac{(2n+1)\binom{2n}{n}}{2^{2n}} = (2n+1)\binom{2n}{n}4^{-n}$$

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- Final solution:

  Closed form for

$$\sum_{k\le n}\binom{2k}{k}4^{-k} = (2n+1)\binom{2n}{n}4^{-n}$$