Discrete Mathematics

Chapter 5 Problem 16
Chapter 4 Problem 14
Evaluate the sum

\[ \sum_{k} \left( \begin{array}{c} 2a \\ a + k \end{array} \right) \left( \begin{array}{c} 2b \\ b + k \end{array} \right) \left( \begin{array}{c} 2c \\ c + k \end{array} \right) (-1)^{k} \]
The binomial coefficient \( \binom{n}{k} \) can be expressed in terms of factorials as follows:

\[
\binom{n}{k} = \frac{n!}{k! \cdot (n - k)!}
\]
Continued...

Let's try to express each of the terms in the problem in factorials:

\[
\binom{2a}{a+k} = \frac{(2a)!}{(2a-(a+k))! \cdot (a+k)!} = \frac{(2a)!}{(a-k)! \cdot (a+k)!}
\]
Similarly,

\[
\binom{2b}{b + k} = \frac{(2b)!}{(b - k)! \cdot (b + k)!}
\]

\[
\binom{2c}{c + k} = \frac{(2c)!}{(c - k)! \cdot (c + k)!}
\]
Therefore,
\[ \sum_{k} \binom{2a}{a+k} \binom{2b}{b+k} \binom{2c}{c+k} (-1)^k \]

\[= \sum (2a)! (2b)! (2c)! (-1)^k \]
\[= \frac{k (a-k)! (a+k)! (b-k)! (b+k)! (c-k)! (c+k)!}{k (a-k)! (a+k)! (b-k)! (b+k)! (c-k)! (c+k)!} \]
Multiplying numerator and denominator by \((a+b)! \cdot (b+c)! \cdot (c+a)!
\)

We will therefore have,

\[
\frac{(2a)! \cdot (2b)! \cdot (2c)!}{(a+b)! \cdot (b+c)! \cdot (c+a)!} \cdot \sum_{k} \frac{(a+b)! \cdot (b+c)! \cdot (c+a)! (-1)^k}{(a-k)! \cdot (a+k)! \cdot (b-k)! \cdot (b+k)! \cdot (c-k)! \cdot (c+k)!}
\]

Constant

Let's try to get a known form for this.
Continued…

Considering:

\[ \sum_{k} \frac{(a+b)! (b+c)! (c+a)!}{(a-k)! (a+k)! (b-k)! (b+k)! (c-k)! (c+k)!} \]

\[ = \sum_{k} \frac{(a+b)! (b+c)! (c+a)!}{(a+k)! (b-k)! (b+k)! (c-k)! (c+k)! (a-k)!} \]

(Just interchanging the order of the terms in the denominator)

We know that,

\[ \frac{(a+b)!}{(a+k)! (b-k)!} = \binom{a+b}{a+k} \]
Similarly,
\[
\frac{(b+c)!}{(b+k)! \ (c-k)!} = \binom{b+c}{b+k}
\]
\[
\frac{(c+a)!}{(c+k)! \ (a-k)!} = \binom{c+a}{c+k}
\]
Therefore,

\[
\sum_{k} \frac{(a+b)! (b+c)! (c+a)!}{(a-k)! (a+k)! (b-k)! (b+k)! (c-k)! (c+k)!} \times (-1)^k
\]

\[
= \sum_{k} \frac{(a+b) (b+c) (c+a)}{(a+k) (b+k) (c+k)} \times (-1)^k
\]

which is a known form.

Using the equation given in Textbook Page No. 171, Eq. 5-29.

We have,

\[
\sum_{k} \frac{(a+b) (b+c) (c+a)}{(a+k) (b+k) (c+k)} \times (-1)^k = \frac{(a + b + c)!}{a! \ b! \ c!}
\]
Thus, the solution for the problem becomes:

\[
\frac{(2a)! (2b)! (2c)!}{(a+b)! (b+c)! (c+a)!} \times \frac{1}{a! b! c!}
\]
Chapter 4, Problem No 14

Does every prime occur as a factor of some Euclid number $e_n$?
Euclid Number:
Definition:

**Euclid numbers** are integers of the form $E_n = p_n\# + 1$,
where $p_n\#$ is the primorial of $p_n$ which is the $n$th prime.
They are named after the ancient Greek mathematician Euclid, who used them in his original proof that there are an infinite number of prime numbers.

Primorial:
For \( n \geq 2 \), the **primorial** \((n#)\) is the product of all prime numbers less than or equal to \( n \). For example, \( 7# = 210 \) is a primorial which is the product of the first four primes multiplied together \((2 \cdot 3 \cdot 5 \cdot 7)\).
The simplest argument could be that to show that there is a prime number which is never the factor of any Euclid number. If we consider any Euclid number, \( p_n\# \) is always a multiple of 2. And Euclid number is 1 added to \( p_n\# \).
Every Euclid number is of the form
\[ = (2 \times k) + 1 \]
where “k” is product of prime numbers \( \leq n \) excluding 2.
So, it is very clear that there exists no Euclid number which is divisible by 2.
Hence, the answer is:
Every prime cannot occur as a factor of some Euclid number $e_n$. 
THANK YOU