Chapter 5 Problem 15, 43

CSE 547 – DISCRETE MATHEMATICS

Chapter 5 Problem 43

Problem

Prove the triple binomial identity (5.28)

$$\sum_k \binom{m-r+s}{k}\binom{n+r-s}{n-k}\binom{r+k}{m+n} = \binom{r}{m}\binom{s}{n},$$

where integers r, s, m, n $\geq$ 0

<u>*Hint:*</u> *First replace* $\binom{r+k}{m+n}$ *by* $\sum_j \binom{r}{m+n-j}\binom{k}{j}$

Formulae used

- Product of two binomial co-efficient

$$\binom{r}{m}\binom{m}{k} = \binom{r}{k}\binom{r-k}{m-k} \qquad m, k \in Z$$

r is real ….. (5.21)

- *Vandermonde's* Convolution

$$\sum_k \binom{r}{m+k}\binom{s}{n-k} = \binom{r+s}{m+n} \qquad m, n \in Z$$

r is real ….. (5.22)

- Symmetry Identity

$$\binom{n}{k} = \binom{n}{n-k} \qquad k, n \in Z \;;\; n >= 0$$

…..(5.4)

$$\sum_k \binom{m-r+s}{k}\binom{n+r-s}{n-k}\binom{r+k}{m+n} = \binom{r}{m}\binom{s}{n}, \quad m, n \in Z \text{ and } m, n \geq 0.$$

Proof

Let us evaluate the L.H.S.

$$\sum_k \binom{m-r+s}{k}\binom{n+r-s}{n-k}\binom{r+k}{m+n}$$

Substituting the given hint

$$\binom{r+k}{m+n} = \sum_j \binom{r}{m+n-j}\binom{k}{j}$$

We get

$$\sum_k \binom{m-r+s}{k}\binom{n+r-s}{n-k} \sum_j \binom{r}{m+n-j}\binom{k}{j}$$

$$\sum_k \binom{m-r+s}{k}\binom{n+r-s}{n-k}\binom{r+k}{m+n} = \binom{r}{m}\binom{s}{n}, \quad m,n \in Z \text{ and } m,n \geq 0.$$

Proof

Rephrasing the summation

$$\sum_k \sum_j \binom{m-r+s}{k}\binom{n+r-s}{n-k}\binom{r}{m+n-j}\binom{k}{j}$$

Interchanging the order of summation (2.27)

$$\sum_j \sum_k \binom{m-r+s}{k}\binom{n+r-s}{n-k}\binom{r}{m+n-j}\binom{k}{j}$$

Note that the term $\binom{r}{m+n-j}$ in the above summation is independent of 'k'. So we rewrite the summation as

$$\sum_j \binom{r}{m+n-j}\sum_k \binom{m-r+s}{k}\binom{n+r-s}{n-k}\binom{k}{j}$$

$$\sum_k \binom{m-r+s}{k}\binom{n+r-s}{n-k}\binom{r+k}{m+n} = \binom{r}{m}\binom{s}{n}, \quad m,n \in Z \text{ and } m,n \geq 0.$$

Proof

Rearranging the terms of the summation

$$\sum_j \binom{r}{m+n-j} \sum_k \binom{m-r+s}{k} \binom{k}{j} \binom{n+r-s}{n-k}$$

We already know from 5.21 $\binom{r}{m}\binom{m}{k} = \binom{r}{k}\binom{r-k}{m-k}$

So the term $\binom{m-r+s}{k}\binom{k}{j}$ of the summation now becomes as

$$\binom{m-r+s}{k}\binom{k}{j} = \binom{m-r+s}{j}\binom{(m-r+s)-(j)}{k-j}$$

$$= \binom{m-r+s}{j}\binom{m-r+s-j}{k-j}$$

$$\sum_k \binom{m-r+s}{k}\binom{n+r-s}{n-k}\binom{r+k}{m+n} = \binom{r}{m}\binom{s}{n}, \quad m,n \in Z \text{ and } m,n \geq 0.$$

Proof

Now our summation becomes

$$\sum_j \binom{r}{m+n-j} \sum_k \binom{m-r+s}{j} \binom{m-r+s-j}{k-j} \binom{n+r-s}{n-k}$$

The term $\binom{m-r+s}{j}$ of the summation is independent of summation variable k, so we have

$$\sum_j \binom{r}{m+n-j} \binom{m-r+s}{j} \sum_k \binom{m-r+s-j}{k-j} \binom{n+r-s}{n-k}$$

$$\sum_k \binom{m-r+s}{k} \binom{n+r-s}{n-k} \binom{r+k}{m+n} = \binom{r}{m} \binom{s}{n}, \quad m, n \in Z \; and \; m, n \geq 0.$$

Proof

Using Vandermonde's equation (5.22) the terms of summation can be simplified as

$$\sum_k \binom{m-r+s-j}{k-j}\binom{n+r-s}{n-k} = \binom{(m-r+s-j)+(n+r-s)}{(k-j)+(n-k)}$$

$$\sum_k \binom{m-r+s-j}{k-j}\binom{n+r-s}{n-k} = \binom{m+n-j}{n-j}$$

$$\sum_k \binom{m-r+s}{k}\binom{n+r-s}{n-k}\binom{r+k}{m+n} = \binom{r}{m}\binom{s}{n}, \quad m,n \in Z \text{ and } m,n \geq 0.$$

Proof

After plugging in the values the summation becomes

$$\sum_j \binom{r}{m+n-j}\binom{m-r+s}{j}\binom{m+n-j}{n-j}$$

Rearranging the terms,

$$\sum_j \binom{r}{m+n-j}\binom{m+n-j}{n-j}\binom{m-r+s}{j}$$

$$\sum_k \binom{m-r+s}{k}\binom{n+r-s}{n-k}\binom{r+k}{m+n} = \binom{r}{m}\binom{s}{n}, \quad m,n \in Z \text{ and } m,n \geq 0.$$

Proof

Transforming the term $\binom{r}{m+n-j}\binom{m+n-j}{n-j}$

$$\binom{r}{m+n-j}\binom{n+m-j}{n-j} = \binom{r}{n-j}\binom{r-n+j}{m}$$ …..by 5.21

$$\binom{r}{n-j}\binom{r-n+j}{m} = \binom{r}{r-n+j}\binom{r-n+j}{m}$$ $(r \geq 0)$ ….by symmetry

$$\binom{r}{r-n+j}\binom{r-n+j}{m} = \binom{r}{m}\binom{r-m}{r-n+j-m}$$ …..by 5.21

$$\binom{r}{m}\binom{r-m}{r-n+j-m} = \binom{r}{m}\binom{r-m}{n-j}$$ ….by symmetry

$$\sum_k \binom{m-r+s}{k}\binom{n+r-s}{n-k}\binom{r+k}{m+n} = \binom{r}{m}\binom{s}{n}, \quad m,n \in Z \text{ and } m,n \geq 0.$$

Proof

Now we have obtained the result

$$\binom{r}{m+n-j}\binom{m+n-j}{n-j} = \binom{r}{m}\binom{r-m}{n-j}$$

Our original summation is

$$\sum_j \binom{r}{m+n-j}\binom{n+m-j}{n-j}\binom{m-r+s}{j}$$

$$= \sum_j \binom{r}{m}\binom{r-m}{n-j}\binom{m-r+s}{j}$$ …using the above result

$$\sum_k \binom{m-r+s}{k}\binom{n+r-s}{n-k}\binom{r+k}{m+n} = \binom{r}{m}\binom{s}{n}, \ m,n \in Z \text{ and } m,n \geq 0.$$

Proof

$$\sum_j \binom{r}{m}\binom{r-m}{n-j}\binom{m-r+s}{j}$$

The term is independent of summation variable j

$$= \binom{r}{m}\sum_j \binom{r-m}{n-j}\binom{m-r+s}{j}$$

Applying Vandermonde's convolution (5.22) on the summation term

$$\sum_j \binom{r-m}{n-j}\binom{m-r+s}{0+j} = \binom{(r-m)+(m-r+s)}{n+0} = \binom{s}{n}$$

$$= \binom{r}{m}\binom{s}{n}$$

$$\sum_k \binom{m-r+s}{k}\binom{n+r-s}{n-k}\binom{r+k}{m+n} = \binom{r}{m}\binom{s}{n},\ \ m,n \in Z \text{ and } m,n \geq 0.$$

Proof

So
$$\sum_j \binom{r-m}{n-j}\binom{m-r+s}{j} = \binom{s}{n}$$

Using the above result
$$\binom{r}{m}\sum_j \binom{r-m}{n-j}\binom{m-r+s}{j} = \binom{r}{m}\binom{s}{n}$$

Thus
$$\sum_k \binom{m-r+s}{k}\binom{n+r-s}{n-k}\binom{r+k}{m+n} = \binom{r}{m}\binom{s}{n}$$

Hence our result

$$\sum_k \binom{m-r+s}{k}\binom{n+r-s}{n-k}\binom{r+k}{m+n} = \binom{r}{m}\binom{s}{n}, \quad m,n \in Z \; and \; m,n \geq 0.$$

Chapter 5 Problem 15

Problem

What is

$$\sum_k \binom{n}{k}^3 (-1)^k \quad ?$$

<u>*Hint:*</u> *Use 5.29*

$$\sum_k \binom{a+b}{a+k}\binom{b+c}{b+k}\binom{c+a}{c+k}(-1)^k = \frac{(a+b+c)!}{a!b!c!}$$

where integers a,b,c $\geq$ 0

Formulae used

- Binomial co-efficient

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} \qquad n, k \in Z \quad n, k >= 0 \qquad \text{….. (5.3)}$$

- Symmetry Identity

$$\binom{n}{k} = \binom{n}{n-k} \qquad k, n \in Z \; ; \; n >= 0 \qquad \text{…..(5.4)}$$

$$Evaluate \quad \sum_k \binom{n}{k}^3 (-1)^k$$

Evaluation

- Let us evaluate the sum $\sum_k \binom{n}{k}^3 (-1)^k$ for small values of n

n	Summation
0	1
1	0
2	-6
3	0
4	90
5	0
6	-1680

<u>n = 0</u> $\binom{0}{0}^3 (-1)^0 = 1$

<u>n = 1</u> $\binom{1}{0}^3 (-1)^0 + \binom{1}{1}^3 (-1)^1 = 0$

<u>n = 2</u> $\binom{2}{0}^3 (-1)^0 + \binom{2}{1}^3 (-1)^1 + \binom{2}{2}^3 (-1)^2 = -6$

<u>n = 3</u> $\binom{3}{0}^3 (-1)^0 + \binom{3}{1}^3 (-1)^1 + \binom{3}{2}^3 (-1)^2 + \binom{3}{3}^3 (-1)^3 = 0$

<u>n = 4</u> $\binom{4}{0}^3 (-1)^0 + \binom{4}{1}^3 (-1)^1 + \binom{4}{2}^3 (-1)^2 + \binom{4}{3}^3 (-1)^3 + \binom{4}{4}^3 (-1)^4 = 90$

$$\textit{Evaluate} \quad \sum_k \binom{n}{k}^3 (-1)^k$$

Evaluation

- We observe that the sum changes according to nature of ‘n’, i.e. either when ‘n’ is even or ‘n’ is odd

- So we evaluate the summation by considering the following cases

  - Case A: n is odd
  - Case B: n is even

$$\text{Evaluate} \quad \sum_k \binom{n}{k}^3 (-1)^k$$

Evaluation: Case A – ‘n’ is odd

- When n is odd we have n+1 (even number) terms in the summation $\sum_k \binom{n}{k}^3 (-1)^k$ from k=0 to k=n

- Expanding the terms of the summation

$$\sum_k \binom{n}{k}^3 (-1)^k = \binom{n}{0}^3 - \binom{n}{1}^3 + \binom{n}{2}^3 - \ldots\ldots\ldots - \binom{n}{n-2}^3 + \binom{n}{n-1}^3 - \binom{n}{n}^3$$

- Applying symmetry identity,

$$\sum_k \binom{n}{k}^3 (-1)^k = \binom{n}{0}^3 - \binom{n}{1}^3 + \binom{n}{2}^3 - \ldots\ldots\ldots - \binom{n}{2}^3 + \binom{n}{1}^3 - \binom{n}{0}^3 = 0$$

$$\textit{Evaluate} \quad \sum_k \binom{n}{k}^3 (-1)^k$$

Evaluation: Case B – 'n' is even

- When n is even we have n+1 (odd number) terms in the summation $\sum_k \binom{n}{k}^3 (-1)^k$ from k=0 to k=n

- We evaluate this sum in a different way, by expansion

$$\sum_k \binom{n}{k}^3 (-1)^k = \sum_k \binom{n}{k}\binom{n}{k}\binom{n}{k}(-1)^k$$

- Substituting 'k' by $\frac{n}{2}+k$ the summation becomes

$$\sum_{\frac{n}{2}+k} \binom{\frac{n}{2}+\frac{n}{2}}{\frac{n}{2}+k}\binom{\frac{n}{2}+\frac{n}{2}}{\frac{n}{2}+k}\binom{\frac{n}{2}+\frac{n}{2}}{\frac{n}{2}+k}(-1)^{\frac{n}{2}+k}$$

$$\textit{Evaluate} \quad \sum_k \binom{n}{k}^3 (-1)^k$$

Evaluation: Case B – 'n' is even

- Substituting 'p' for $\frac{n}{2}$ the summation now becomes

$$\sum_{\frac{n}{2}+k} \binom{\frac{n}{2}+\frac{n}{2}}{\frac{n}{2}+k}\binom{\frac{n}{2}+\frac{n}{2}}{\frac{n}{2}+k}\binom{\frac{n}{2}+\frac{n}{2}}{\frac{n}{2}+k}(-1)^{\frac{n}{2}+k} = \sum_{p+k}\binom{p+p}{p+k}\binom{p+p}{p+k}\binom{p+p}{p+k}(-1)^{p+k}$$

where

$$0 \le p+k \le n$$

Subtracting p in the above predicate $-p \le k \le n-p$

$$\because \; p = \frac{n}{2} \;\Rightarrow\; n = 2p \qquad -p \le k \le p$$

But by the formal definition of binomial co-efficient $\binom{n}{k} = 0 \; for \; k < 0$

Thus $0 \le k \le p$

$$Evaluate \;\; \sum_k \binom{n}{k}^3 (-1)^k$$

Evaluation: Case B – ‘n’ is even

- With the new limit for summation variable ‘k’ $0 \le k \le p$

the summation now becomes

$$\sum_k \binom{p+p}{p+k}\binom{p+p}{p+k}\binom{p+p}{p+k}(-1)^{p+k}$$

$$= \sum_k \binom{p+p}{p+k}\binom{p+p}{p+k}\binom{p+p}{p+k}(-1)^{p}(-1)^{k}$$

The term $(-1)^p$ is independent of summation variable ‘k’. So we have,

$$= (-1)^p \sum_k \binom{p+p}{p+k}\binom{p+p}{p+k}\binom{p+p}{p+k}(-1)^{k}$$

$$\textit{Evaluate} \quad \sum_k \binom{n}{k}^3 (-1)^k$$

Evaluation: Case B – ‘n’ is even

Using the given hint : $$\sum_k \binom{a+b}{a+k}\binom{b+c}{b+k}\binom{c+a}{c+k}(-1)^k = \frac{(a+b+c)!}{a!b!c!}$$

Our summation becomes,

$$(-1)^p \sum_k \binom{p+p}{p+k}\binom{p+p}{p+k}\binom{p+p}{p+k}(-1)^k$$

$$= \frac{(p+p+p)!}{p!\,p!\,p!}(-1)^p$$

$$i.e.\ \sum_k \binom{n}{k}^3 (-1)^k = \frac{(3p)!}{(p!)^3}(-1)^p$$

$$Evaluate \quad \sum_k \binom{n}{k}^3 (-1)^k$$

Correctness of the Solution

- Lets check the result when n is ‘even’

for small values of n with our prior values

$$n = 0 \Rightarrow p = 0 \ \therefore \ \frac{(3p)!}{(p!)^3}(-1)^p = \frac{(0)!}{(0!)^3}(-1)^0 = 1$$

$$n = 2 \Rightarrow p = 1 \ \therefore \ \frac{(3p)!}{(p!)^3}(-1)^p = \frac{(3)!}{(1!)^3}(-1)^1 = -6$$

$$n = 4 \Rightarrow p = 2 \ \therefore \ \frac{(3p)!}{(p!)^3}(-1)^p = \frac{(6)!}{(2!)^3}(-1)^2 = 90$$

- These values match with our prior computation for n

$$Evaluate \quad \sum_k \binom{n}{k}^3 (-1)^k$$

Result

$$\sum_k \binom{n}{k}^3 (-1)^k \quad = \ 0 \qquad\qquad if\ \ n\ is\ odd$$

$$\sum_k \binom{n}{k}^3 (-1)^k = \frac{(3p)!}{(p!)^3}(-1)^p \qquad if\ \ n\ is\ even$$

$$where \quad p = \frac{n}{2}$$

$$Evaluate \quad \sum_k \binom{n}{k}^3 (-1)^k$$