CHAPTER 5

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BINOMIAL COEFFICIENTS

DEF 1

$$\binom{n}{k} = \frac{n(n-1)\dots(n-k+1)}{k(k-1)\dots 2\cdot 1} = \frac{n^{\underline{k}}}{k!}$$

$$\boxed{\binom{n}{k} = \frac{n^{\underline{k}}}{k!}} \quad \text{FOR} \quad \boxed{k \geq 0,\ n-k \in \mathbb{N}, \quad k \leq n}$$

COMBINATORIAL INTERPRETATION

$\binom{n}{k}$ - "n choose k"

$\binom{n}{k}$ denotes a NUMBER of ways to choose K-ELEMENT SUBSE from an n-element SET.

PROOF of COMBINATORIAL STATEMENT

$\binom{n}{k} = \frac{n^{\underline{k}}}{k!}$ is a number of ways to choose k-elem SUBSETS from n-elem SET.

STEP 1 Find the number of k-element (1-1) sequences formed out of n-element SET

REMARK 1 - all sequences of lenght k from n-elem set are all functions

$f: \{1,..k\} \to \{a_1,..a_n\}$

and there are n^k of them.

We need 1-1 sequences only.

REMARK 2

DEF PERMUTATION of $A = \{a_1,..a_n\}$, $|A| = n$ is any FUNCTION $f: A \xrightarrow[\text{onto}]{1-1} A$

We prove:

If $|A| = n$, then # of PERMUTATIONS of A is $n!$ $n \geq 0$

PROOF of Remark 2

If $|A|=n$, then there are $n!$ functions f. $f: A \xrightarrow[\text{onto}]{1\text{-}1} A$, $n \geq 1$

By induction over $|A|=n$

$n=1$ $A=\{a\}$ and there is one function $f: \{a\} \xrightarrow[\text{onto}]{1\text{-}1} \{a\}$, $f(a)=a$ and $1!=1$ yes

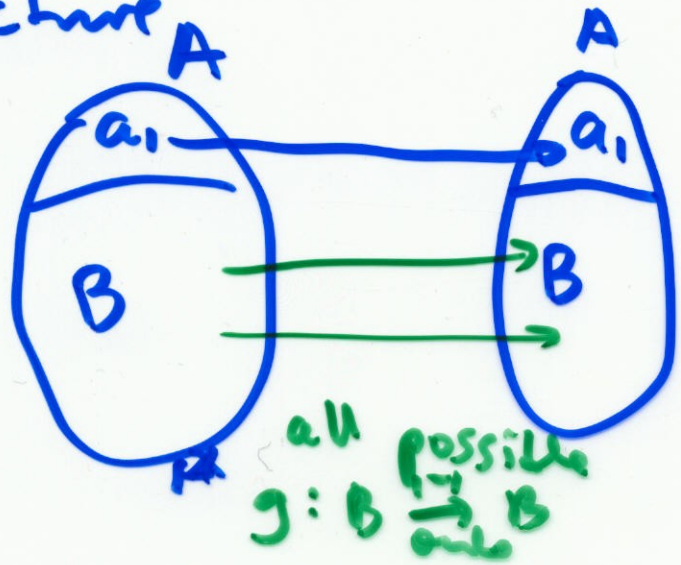
Assume that $B \subseteq A$ (any subset of A) and $|B|=n-1$, then there is $(n-1)!$ functions that map $f: B \xrightarrow[\text{onto}]{1\text{-}1} B$

We count $f: A \xrightarrow[\text{onto}]{1\text{-}1} A$ as follows.

Assume $A=\{a_1, a_2 \ldots a_n\}$

$f(a_1)=a_1$ Fixed a_1 By ind assumption there are $(n-1)!$ such functions

Picture



$|B|=n-1$

$f: A \xrightarrow[\text{onto}]{1\text{-}1} A$

$f(a_1)=a_1$, $f(b)=g(b)$ for $b \in B$

$g(b)=b'$

Group 2

$f(a_2)=a_2$ Fix a_2 + $f(b)=g(b)$

for $b \in B=\{a_1, a_3 .. a_n\}$ $|B|=n-1$

There are $(n-1)!$ functions in G2

Group i $i \geq 2$

$f(a_i)=a_i$ + $f(b)=g(b)$

for $B=A-\{a_i\}=B$ $g: B \xrightarrow[\text{onto}]{1\text{-}1} B$

There are $(n-1)!$ functions

There are n groups; all together there are

$n \cdot (n-1)!$ functions $f: A \xrightarrow[\text{onto}]{1\text{-}1} A$

$n! = n(n-1)!$ end of proof of RETARLE 2

In particular $A=\{a_1 .. a_k\}$

There are $k!$ 1-1 sequences of length k out of A.

BACK TO STEP ONE

Let $|A| = n$ be an n-element set

We construct all 1-1 k-element sequences out of elements of A as follows: sequence $b_1 b_2 \dots b_k$ 1-1, all $i \neq j$, $b_i \neq b_j$

① b_1 – there are n choices, $b_1 \in A$

② $b_1 b_2$ – there are $n-1$ choices $\neq b_1$

③ $b_1 b_2 b_3$ – there are $n-2$ choices for $b_3 \neq b_2 \neq b_1$, $b_i \in A$

Induction (really)

$b_1 b_2 \dots b_k$ – there are $(n-k+1)$ choices for $b_k \neq b_{k-1} \dots \neq b_1$

All together we have

$$n(n-1)\dots(n-k+1)$$

possible 1-1 sequences $b_1 b_2 \dots b_k$

STEP 2 $\binom{n}{k}$ represents how many are there k-element SUBSETS

We know that there are $n(n+1)..(n+k+1)$ sequences 1-1

$b_1 b_2 \ldots b_k$ ⟷ ? ⟷ $\{b_1 b_2 .. b_k\}$

1-1 sequence — subset

{ SETS: $\{b_1 b_2 b_3 .. b_k\} = \{b_2 b_1 b_3 .. b_k\}$ etc

different SEQUENCES represent the same SET

How many of all possible "representations" – as many as PERMUTATIONS i.e $k!$

$$\binom{n}{k} = \frac{\#\text{sequences}}{k!} = \frac{n(n+1)..(n+k+1)}{k!}$$

END

GENERALIZATION

We proved (COMBINATORIAL INT) $\binom{n}{k} = \frac{n^{\underline{k}}}{k!}$ is a number of k-elem subsets of an n-elem set.

$k, n \in \mathbb{N}$ $k \le n$

$0! = 1$

$x^{\underline{0}} = 1$

$\binom{0}{0} = \frac{0^{\underline{0}}}{1} = 1$

$\binom{n}{n} = \frac{n^{\underline{n}}}{n!} = \frac{n!}{n!} =$

$n^{\underline{n}} = n!$

$1^{\underline{n}} = n!$

We had $f : \mathbb{R} \to \mathbb{R}$

$f(x) = x^{\underline{k}} = x(x-1)\dots(x-k+1)$

or $f : \mathbb{R} \times \mathbb{N} \to \mathbb{R}$ (really)

$f(x,k) = x^{\underline{k}}$

$f : \mathbb{R} \to \mathbb{R}$ for any $k \in \mathbb{R}$

We DEFINE $f : \mathbb{R} \times \mathbb{Z} \to \mathbb{R}$

or even $f : \mathbb{C} \times \mathbb{Z} \to \mathbb{R}$

$$f(x,k) = \begin{cases} \frac{x^{\underline{k}}}{k!} & k \ge 0 \\ 0 & k < 0 \end{cases}$$

$\mathbb{C}$ - complex numbers

$f(x,k) = \binom{x}{k} = g(x)$ — for any $k \in \mathbb{Z}$

EXAMPLE

$$\binom{x}{k} = \frac{x^{\underline{k}}}{k!} \qquad k \geq 0$$

$$\binom{-1}{3} = \frac{(-1)^{\underline{3}}}{3!} = \frac{(-1)(-2)(-3)}{1 \cdot 2 \cdot 3} = -1 \qquad k < 0 \Rightarrow = 0$$

$$x^{\underline{k}} = x(x-1)\cdots(x-k+1)$$

$$\binom{n}{n} = 1 \qquad n \in \mathbb{N}$$

$$\binom{-1}{-1} = 0 \quad (k<0) \; !$$

$$\binom{1}{1} = 1 \quad \text{in general} \quad \binom{n}{n} = 1 \qquad n \geq 0$$

$$\binom{n}{n} = 0 \quad n<0 \qquad \text{only when } n \geq 0$$

$$\binom{\sqrt{2}}{3} = \frac{(\sqrt{2})^{\underline{3}}}{3!} = \frac{\sqrt{2}(\sqrt{2}-1)(\sqrt{2}-2)}{6}$$

$$(\sqrt{2})^{\underline{3}} = \sqrt{2}(\sqrt{2}-1)(\sqrt{2}-2)$$

NO COMBINATORIAL INTERPRETATION!

We defined $f: R \times Z \to \mathbb{R}$

$$\binom{x}{k} = \begin{cases} \frac{x^{\underline{k}}}{k!} & k \geq 0 \\ 0 & k < 0 \end{cases}$$

where $x \in R$, $k \in Z$

$f(x,k) = \binom{x}{k}$

$f: R \times Z \to R$

$f(x,k) = \binom{x}{k}$

$k! = 1^{\underline{k}}$ $k \in$

We use (x) to denote that or

UPPER LIMIT is a real number

$k! = k^{\underline{k}}$

We use $\longrightarrow$ $n \in \mathbb{N}$, $k \in N$

$f: \mathbb{N} \times N \to R$

$$\binom{n}{k} = \frac{n^{\underline{k}}}{k!}$$

FOR COMBINATORIAL interpretation

BOOK USES $r \in R$ instead $x \in R$

$$\binom{r}{k} = \begin{cases} \frac{r^{\underline{k}}}{k!} & k \geq 0 \\ 0 & k < 0 \end{cases}$$

$f(r,k) = \binom{r}{k}$

$f: R \times Z \to R$

SAME FUNCTION

$$\binom{x}{k} = \frac{x^{\underline{k}}}{k!} \quad k \geq 0, \qquad \binom{x}{k} = 0 \quad k < 0$$

$k \in \mathbb{Z}$, $x \in \mathbb{R}$

REMINDER

① $\binom{n}{n} = 1 \quad n \in \mathbb{N}$

② $\binom{n}{m} = 0 \quad n < 0$

③ $\binom{n}{k} = 0$ for $k > n$, $k \geq 0$

WARNING we have restriction $k \geq 0$ in

$$\binom{x}{k} = \frac{x^{\underline{k}}}{k!}$$

i.e $\binom{x}{k} = 0 \quad k < 0$

Let's look at

SYMMETRY IDENTITY

CONSIDER $f : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$

$0 \leq k \leq n$

$$\binom{n}{k} = \frac{n(n+1)\ldots(n-k+1)}{k!} = \frac{n!}{k!\,(n-k)!}$$

$$= \frac{n!}{(n-(n-k))!\,(n-k)!} = \binom{n}{n-k}$$

↑ proof

SYMMETRY PROPERTY

$$\boxed{\binom{n}{k} = \binom{n}{n-k}}$$

for $k, n \in \mathbb{N}$, $k \leq n$

COMBINATORIAL INTERPRETATION

$\binom{n}{k}$ k chosen elements out of n

$\binom{n}{n-k}$ $n-k$ "unchosen" elements out of n.

SYMMETRY PROPERTY C.D

$$\boxed{\binom{n}{k} = \binom{n}{n-k}}$$

Proved $k, n \in$

$k \leq n$

restrictions

$n \in N$

CASE $k < 0$

$$\binom{n}{k} = 0 \quad , \quad \binom{n}{n-k} = \binom{n}{s} = 0$$

$s > n$

CASE $k > n$

$$\binom{n}{k} = 0 \quad , \quad \binom{n}{n-k} = \binom{n}{s} = 0$$

$s < 0$

So we have

SYMMETRY GENERAL

$$\boxed{\binom{n}{k} = \binom{n}{n-k} \qquad n \in N \ , \ k \in Z}$$

$f : N \times Z \to N$

why we can't have $n \in Z$?

Check $\binom{n}{k} = \binom{n}{n-k}$ for $n = -1$

$k \geq 0$ (L)

$$\binom{-1}{k} = \frac{(-1)^{\underline{k}}}{k!} = \frac{(-1)(-2)\ldots(-1-k+1)}{k!}$$ ← k factors

$$= \frac{(-1)^{(k)}\, k!}{k!} = (-1)^k$$

$x^{\underline{k}} = x(x-1)\ldots(x-k+1)$

(R)

$$\binom{-1}{-1-k} = 0 \quad \text{all } k \geq 0$$

By induction.

$$\binom{-1}{k} \pm \binom{-1}{-1-k} \quad \text{all } k \geq 0.$$

ABSORPTION IDENTITY

$$\binom{x}{k} = \frac{x}{k}\binom{x-1}{k-1}$$

$k \neq 0$

$x \in \mathbb{R}$

$k \in \mathbb{Z}^+ \cup \{0\}$

PROOF OF ABSORPTION IDENTITY

① $\binom{x}{k} = \frac{x}{k}\binom{x-1}{k-1}$ $\quad x \in \mathbb{R},\ k \neq 0,\ k \in \mathbb{Z}$

iff

② $k\binom{x}{k} = x\binom{x-1}{k-1}$ $\quad x \in \mathbb{R},\ k \in \mathbb{Z}$

Proof:

① Observe $\quad x^{\underline{k}} = x(x-1)^{\underline{k-1}}$

② $x^{\underline{k}} = x(x-1)\cdots(x-k+1)$

GENERAL $\quad x^{\underline{m+n}} = x^{\underline{m}}(x-m)^{\underline{n}}$

② $x(x-1)^{\underline{k-1}} = x\underbrace{(x-1)(x-2)\cdots((x-1)-(k-1)+1)}$

$= x(x-1)\cdots(x-1-k+1+1)$

Evaluate: $= x(x-1)\cdots(x-k+1) = x^{\underline{k}}$

$\binom{x}{k} = \frac{x(x-1)^{\underline{k-1}}}{k\cdot(k-1)!} = \frac{x}{k}\binom{x-1}{k-1}$ end

ABSORPTION IDENTITY

PROVED

(2) $$k\binom{x}{k} = x\binom{x-1}{k-1}$$

$x \in R$
$k \in Z$

TO BE PROVED:

(3) $$(x-k)\binom{x}{k} = x\binom{x-1}{k}$$

BIG DOMAIN

$x \in R$
$k \in Z$

$L = P$

$N \subseteq R$

Proof

① Prove ~~Assume~~ the case $x \in N$, $k \in Z$

$$L = (x-k)\binom{x}{k} \overset{\text{Symmetry}}{=} (x-k)\binom{x}{x-k}$$

use (2) for $k := x-k$

① $\binom{n}{k} = \binom{n}{n-k}$ SYMMETRY

② $$= x\binom{x-1}{x-k-1}$$

$$= x\binom{\overbrace{x-1}^{n}}{\underbrace{(x-1)-k}_{n-k}} \overset{\text{Symmetry}}{=} x\binom{x-1}{k} = P$$

$\binom{n}{k}$

½ end

We proved identity (3) for a CASE $x \in \mathbb{N}$, $k \in \mathbb{Z}$.

Now we are going to show that this results EXTENDS to $x \in \mathbb{R}$ ($k \in \mathbb{Z}$)

We do it by what is called

A POLYNOMIAL argument.

OBSERVE:

$$(x-1)\binom{x}{k} = x\binom{x-1}{k} \qquad x \in \mathbb{R},\ k \in \mathbb{Z}$$

is EQUALITY OF TWO POLYNOMIALS OF DEGREE $(k+1)$

$$L(x) = (x-1)\binom{x}{k} = a_{k+1}x^{k+1} + \dots + a_0$$

$$P(x) = x\binom{x-1}{k} = b_{k+1}x^{k+1} + \dots + b_0$$

$$\binom{x}{k} = \frac{x(x-1)\dots(x-k+1)}{k(k-1)!}$$

polynomial of degree k over $\mathbb{Z}$.

POLYNOMIALS THEOREMS

THM 1

Let $w(x) = a_n x^n + \dots + a_0$ $\quad a_n \neq 0$

be a polynomial of the degree n

The equation

$$w(x) = 0$$

has at most n solutions; i.e

$$|\{x : w(x) = 0\}| \leq n.$$

THM 2

If $w(x) = a_n x^n + \dots \quad a_0$ $\quad a_n \neq 0$

is such that

$$|\{x : w(x) = 0\}| > n$$

then $w(x) = 0$ for all $x \in R$

BACK TO OUR IDENTITY

$$L(x) = (x-1)\binom{x}{k} = a_{k+1}x^{k+1} \cdots + a_0$$

$$P(x) = x\binom{x-1}{k} = b_{k+1}x^{k+1} + \cdots b_0$$

(3) $\boxed{L(x) = P(x)}$ for all $x \in \mathbb{R}$, $k \in K$

$\updownarrow$

GOAL: all $x \in \mathbb{R}$, $k \in K$

$$\underbrace{L(x) - P(x) = 0}_{w(x)}$$

We have a polynomial

$w(x) = L(x) - P(x)$ of degree $k+1$

$\boxed{|\{x : L(x) - P(x) = 0\}| = x_0 > k+1}$ $(x_0 \in \mathbb{R})$

because we proved that $L(x) = P(x)$ for all $x \in \mathbb{N}$.

we proved $\forall x \in \mathbb{R}$

by Thm 2 $w(x) = L(x) - P(x) = 0$ for all $x \in \mathbb{R}$.

HENCE

$L(x) = P(x)$, for all $x \in \mathbb{R}$, $k \in K$.

$$(x-k)\binom{x}{k} \stackrel{?}{=} x\binom{x-1}{k} \quad (3)$$

Independent proof Ba

$$x\binom{x}{k} - k\binom{x}{k} \stackrel{?}{=} x\binom{x-1}{k}$$

(2)

$$x\binom{x}{k} - x\binom{x-1}{k-1} \stackrel{?}{=} x\binom{x-1}{k}$$

$$x\binom{x}{k} \stackrel{?}{=} x\binom{x-1}{k} + x\binom{x-1}{k-1}$$

$$x\binom{x}{k} \stackrel{?}{=} x\left(\binom{x-1}{k} + \binom{x-1}{k-1}\right)$$

$$\binom{x-1}{k} \stackrel{?}{=} \frac{(x-1)(x-2)\cdots\cdots(x-1-k+1)}{k!}$$

$$x-1-k+1 = (x-1)(x-2)\cdots\cdots(x-k)$$

$$k\cdot(k-1)!$$

$$\binom{x-1}{k-1} = \frac{(x-1)\cdots\cdots(x-k+1)}{(k-1)!}$$

$$\frac{(x-1)^{k-1}}{(k-1)!}\left(\frac{1}{k} + \frac{x-k}{k}\right)$$

$$\left(1 + \frac{x}{k} - 1\right)$$

$$x\binom{x}{k} \overset{?}{=} \binom{x-1}{k-1} \cdot \frac{x}{k} \cdot x$$

$x \neq 0$

$$\binom{x}{k} \overset{?}{=} \binom{x-1}{k-1} \frac{x}{k}$$

$$k\binom{x}{k} = x\binom{x-1}{k-1}$$ yes was proved

+ check case

$x = 0$

SYMMETRY PROPERTY

① $\binom{n}{k} = \binom{n}{n-k}$ $n \in \mathbb{N}$ $k \in \mathbb{Z}$

ABSORBTION IDENTITY

② $\binom{n}{k} = \frac{n}{k}\binom{n-1}{k-1}$ $n \in \mathbb{R}$, $k \in \mathbb{Z} - \{0\}$

③ $k\binom{n}{k} = n\binom{n-1}{k-1}$ $n \in \mathbb{R}$ $k \in \mathbb{Z}$

PROVE FIRST FOR $n \in \mathbb{N}$, and then extend it via POLYNOMIAL ARGUMENT to $n \in \mathbb{R}$

④ $\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$ $n \in \mathbb{R}$, $k \in \mathbb{R}$

$$\binom{n}{k} = \frac{n^{\underline{k}}}{k!} = \frac{n(n-1)\cdots(n-k+1)}{k!}$$

when $n \in \mathbb{R}$ we put

$k \geq 0$, $k \leq n$, $n, k \in \mathbb{N}$; 0 $k < 0$

NEXT IDENTITY

(4) $$\binom{x}{k} = \binom{x-1}{k} + \binom{x-1}{k-1}$$ $x \in \mathbb{R}$, $k \in \mathbb{Z}$

CASE 1 Assume $x \in \mathbb{N}$, i.e $n \in \mathbb{N}$

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$$

and we have a COMBINATORIAL INTERPRETATION:

$\binom{n}{k}$ = # of k-element subsets chosen from the n-elem. set

If we have n-eggs with exactly one bad egg, we have

$\binom{n-1}{k}$ selections that have good eggs

$\binom{n-1}{k-1}$ of them contain the BAD egg because they have $k-1$ of $n-1$ good eggs.

PROVED

so

$$\boxed{\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}} \qquad n \in \mathbb{N},\ k \in \mathbb{Z}$$

all eggs — only good — contain bad egg

+ POLYNOMIAL ARGUMENT

$L(x) = \binom{x}{k}$ polynomial of the degree k $\quad x \in \mathbb{R}$

$P(x) = \binom{x-1}{k} + \binom{x-1}{k-1}$ polynomial of the degree k

we proved

$L(x) - P(x) = 0$ (polyn of degree k) for all $x \in \mathbb{N}$

by thm 2

$L(x) - P(x) = 0$ for all $x \in \mathbb{R}$, $k \in \mathbb{N}$

$\boxed{L(x) = P(x) \quad \text{for all } x \in \mathbb{R},\ k \in \mathbb{N}}$

PROOF 2

Use identities ③ + ②

$x \in \mathbb{R}$
$k \in \mathbb{Z}$

We know

③ $(x-k)\binom{x}{k} = x\binom{x-1}{k}$,

② $k\binom{x}{k} = x\binom{x-1}{k-1}$

ADD THEM : ③ + ②

$$(x-k)\binom{x}{k} + k\binom{x}{k} = x\binom{x-1}{k} + x\binom{x-1}{k-1}$$

$$\binom{x}{k}(x-k+k) = x\left(\binom{x-1}{k} + \binom{x-1}{k-1}\right)$$

We get:

$$\binom{x}{k}x = x\left(\binom{x-1}{k} + \binom{x-1}{k-1}\right) \quad * \quad x \neq 0$$

$$\boxed{\binom{x}{k} = \binom{x-1}{k} + \binom{x-1}{k-1}} \quad \text{check } x=0$$

Check CASE $k=0$

$$\binom{0}{k} = \binom{-1}{k} + \binom{-1}{k-1}$$ 2.

① $k<0$ we: $\binom{x}{k} = 0$ for all $k<0$

$L = \mathbb{R}$

$0 = 0 + 0$

$$\binom{-1}{k} = \frac{(-1)^k}{k!} \quad \text{for } k \geq 0, \quad = 0 \text{ for } k<$$

$$\binom{-1}{k-1} = \frac{(-1)^{k-1}}{(k-1)!} \quad \text{for } k \geq 1, \quad = 0 \text{ for } k<1$$

$x^0 =$

② $k=0$

$$\binom{0}{0} \stackrel{?}{=} \binom{-1}{0} + \binom{-1}{-1}$$

$$1 = 1 + 0$$

$$\binom{-1}{0} = \frac{(-1)^0}{1} = 1$$

$$\binom{n}{k} = 0 \quad k>n$$

$$\binom{n}{m} = 0, \quad m<0$$

③ $K>0$; $K \geq 1$

$$\binom{0}{k} = \binom{-1}{k} + \binom{-1}{k-1}$$

$k-1<1$
$k<1$

$$\binom{-1}{k} = \frac{(-1)^{k}}{k!}$$

$$\binom{-1}{k-1} = \frac{(-1)^{k-1}}{(k-1)!}$$

L =

$$\binom{0}{k} = \frac{0^{k}}{k!} = 0$$

use

$$x^{m+n} = x^{m}(x-m)^{n}$$

Evaluate

P =

$$\binom{-1}{k} + \binom{-1}{k-1} =$$

$$= \frac{(-1)^{k}}{k!} + \frac{(-1)^{k-1}}{(k-1)!}$$

$$= \frac{(-1)^{k-1}(-k)}{(k-1)!\,k} + \frac{(-1)^{k-1}}{(k-1)!}$$

$$= \frac{(-1)^{k-1}}{(k-1)!}\left((-1)+1\right) = 0$$

$$(-1)^{k} = (-1)^{(k-1)+1}$$

$$= (-1)^{k-1}(-1)^{1}$$

$$= (-1)^{k-1}(-k)$$

$L = P$.

Proof 3

$$\boxed{\binom{x-1}{k} + \binom{x-1}{k-1}} \overset{\text{def}}{=} \frac{(x-1)^{\underline{k}}}{k!} + \frac{(x-1)^{\underline{k-1}}}{(k-1)!}$$

$$= \frac{(x-1)^{\underline{k-1}}(x-k)}{k!} + \frac{(x-1)^{\underline{k-1}} \cdot k}{k!}$$

$$= \frac{(x-1)^{\underline{k-1}}}{k!}\,(x-k+k)$$

$$= \frac{x \cdot (x-1)^{\underline{k-1}}}{k!} = \frac{x^{\underline{k}}}{k!} = \boxed{\binom{x}{k}}$$

$k \geq 0$, $k-1 \geq 0$, $k \geq 1$

OBSERVE

$$\binom{5}{3} = \binom{4}{3} + \binom{4}{2}$$

$$= \binom{4}{3} + \binom{3}{2} + \binom{3}{1}$$

$$= \binom{4}{3} + \binom{3}{2} + \binom{2}{1} + \underbrace{\binom{2}{0}}$$

$$= \binom{4}{3} + \binom{3}{2} + \binom{2}{1} + \binom{1}{0} + \underbrace{\binom{1}{-1}}_{=0} \quad \text{STOP}$$

GENERAL (proof by induction)

$$\sum \binom{x+k}{k} = \binom{x}{0} + \binom{x+1}{1} + \cdots \binom{x+n}{n} = \binom{x+n+1}{n}$$

$0 \le k \le n$

$n \in \mathbb{Z}$

when $k < 0$ factors are 0. GUESS

We guessed

① $$\sum_{k \le n} \binom{x+k}{k} = \binom{x+n+1}{n}$$

$n \in \mathbb{Z}$
$k \in \mathbb{Z}$
$x \in \mathbb{R}$

We use now formula

$$\binom{x}{k} = \binom{x-1}{k} + \binom{x-1}{k-1}$$

to unfold (differently than before)

$$\underset{n+1}{\binom{5}{3}} = \binom{4}{3} + \binom{4}{2} = \underset{\sim}{\binom{3}{3}} + \binom{3}{2} + \binom{4}{2}$$

$n+1$ ↓ unfold; $\sim$ unfold

$$= \underset{\sim}{\binom{2}{3}} + \binom{2}{2} + \binom{3}{2} + \binom{4}{2}$$

$$= \underset{\sim}{\binom{1}{3}} + \binom{1}{2} + \binom{2}{2} + \binom{3}{2} + \binom{4}{2}$$

$$= \binom{0}{3} + \binom{0}{2} + \binom{1}{2} + \binom{2}{2} + \binom{3}{2} + \binom{4}{2}$$

n ↓ m

and we guess a new formula:

$$\sum \binom{k}{m} \qquad m = 2$$

We guess

n CONSTANT

(2) $$\sum_{0 \le k \le n} \binom{k}{m} = \binom{0}{m} + \binom{1}{m} + \cdots \binom{n}{m} = \binom{n+1}{m+1}$$

need $k \ge 0$

for $k < 0$ same

Both guesses (1) and (2) can $\binom{k}{m} \neq 0$

be proved by math induction

We will prove (2) by induction

and then we will prove (1) from (2):

combinatorial interpretation of (2):

IF we want to choose $(m+1)$ tickets

from a set of $(n+1)$ tickets

numbered $0, 1, \ldots n$

there are $\binom{K}{m}$ ways of doing this

when the largest ticket selection

number is K

$$\binom{0}{m} + \binom{1}{m} + \binom{2}{m} + \cdots \binom{n}{m} = \binom{n+1}{m+1}$$

K=0 K=1

Proof by induction of (2)

$$\sum_{k=0}^{n} \binom{k}{m} = \binom{n+1}{m+1}$$

$n, k, m \in \mathbb{N}$

Induction over $n \in \mathbb{N}$

$n=0$ $\quad \sum_{k=0}^{0} \binom{k}{m} = \binom{0}{m} \overset{?}{=} \binom{1}{m+1}$ $\quad \binom{0}{0} = 1$

$$\binom{1}{m+1} = \binom{0}{m+1} + \binom{0}{m} = 0 \quad (*), \quad \binom{0}{m} = 0 \quad m > 0$$

IND $\quad m > 0$

Assume

$$S_{n-1} = \binom{n}{m+1}$$

Prove

$$S_n = \sum_{k=0}^{n} \binom{k}{m} = \binom{n+1}{m+1}$$

$k \in \mathbb{Z}$

use:

$$\binom{x}{k} = \binom{x-1}{k} + \binom{x-1}{k-1}$$

$$S_n = S_{n-1} + \binom{n}{m}$$

$$\binom{n+1}{m+1} \overset{?}{=} \binom{n}{m+1} + \binom{n}{m}$$

$\overset{\text{Ind}}{=}$

$x = n+1$

$k = m+1$

$n = x-1$

$k-1 = m$

$$\binom{n+1}{m+1} = \binom{n}{m+1} + \binom{n}{m}$$

yes.

(*) $m=0$ $\quad \binom{1}{1} = 1 = \binom{0}{0} = 1$

PROOF of ① from ②

$m \in \mathbb{N}$ $n \in \mathbb{Z}$

① $$\sum_{k \le n} \binom{m+k}{k} = \binom{m+n+1}{n}$$

② $$\sum_{k=0}^{m} \binom{k}{m} = \binom{m+1}{m+1}$$

$k, m \in \mathbb{N}$

USE $$\binom{n}{k} = \binom{n}{n-k}$$ (x) $n \in \mathbb{N}$, $k \in \mathbb{Z}$

Evaluate for $m+k \ge 0$, $k \ge -m$ (out but terms are 0), $k < -m$

$$\sum_{k \le n} \binom{m+k}{k} =$$

$$= \sum \binom{m+k}{k}$$

$$= \sum_{-m \le k \le n} \binom{m+k}{m} \overset{(x)}{=} \sum_{k=0}^{n+m} \binom{k}{m}$$

Replace $k \to k-m$

$-m \le k \le n$

$-m \le k-m \le n$

$0 \le k \le n+m$

$$\overset{②}{=} \binom{n+m+1}{m+1} \overset{(x)}{=} \binom{n+m+1}{n}$$

$n+m+1-(m+1) = n$

$n - k$

Consider ① SPECIAL CASES

$$\sum_{k=0}^{n} \binom{k}{m} = \binom{n+1}{m+1}$$ PROVED

$m, n \in \mathbb{N}$

Consider $m = 1$

$$\binom{n+1}{2} = \sum_{k=0}^{n} \binom{k}{1} = \binom{0}{1} + \binom{1}{1} + \binom{2}{1} + \cdots \binom{n}{1}$$

$$= 0 + 1 + 2 + \cdots n = \frac{n(n+1)}{2}$$

General case $\binom{k}{m}$:

$$\sum_{k=0}^{n} \binom{k}{m} = \sum_{k=0}^{n} \frac{k^{\underline{m}}}{m!} = \frac{1}{m!} \sum_{k=0}^{n} k^{\underline{m}}$$

$\binom{n+1}{m+1}$ "

$$= \binom{n+1}{m+1} \quad \therefore e$$

$$\sum_{k=0}^{n} k^{\underline{m}} = m! \binom{n+1}{m+1}$$ FORMULA

verified C.d

$$\sum_{k=0}^{n} k^{\underline{m}} = m!\binom{n+1}{m+1} = \frac{n^{\underline{k}}}{k!}$$

$$= \frac{m!\,(n+1)^{\underline{m+1}}}{(m+1)!}$$

$$= \frac{(n+1)^{\underline{m+1}}}{m+1}$$

SPECIAL SUM.

$$\sum_{k=0}^{n} k^{\underline{m}} = \frac{(n+1)^{\underline{m+1}}}{(m+1)}$$

$n, m \in \mathbb{N}$

YOU CAN GET IT BY INTEGRATION (short)

Look back at

$$\binom{x}{k} = \binom{x-1}{k} + \binom{x-1}{k-1}$$ - re-write it as $x = x+1$

$$\binom{x+1}{m} = \binom{x}{m} + \binom{x}{m-1}$$ (x) and evaluate

$$\Delta\left(\binom{x}{m}\right) = \binom{x+1}{m} - \binom{x}{m} = \binom{x}{m-1}$$

DEFINITION

↘ from (x)

$$\Delta\left(\binom{x}{m}\right) = \binom{x}{m-1} \qquad x \in \mathbb{R},\ m \in \mathbb{Z}$$

and "integral"

$$\sum \binom{x}{m} \delta x = \binom{x}{m+1} + C$$

We get a SHORT PROOF of (2):

$$\sum_{x=0}^{n} \binom{x}{k} \delta x = \binom{x}{k+1}\Big|_{0}^{n+1} = \binom{n+1}{k+1} - \binom{0}{k+1}$$

$$= \binom{n+1}{k+1}$$

Used

$$\sum_{k=a}^{b-1} g(k) = \sum_{a}^{b} g(x)\,\delta x$$