# NEGATING UPPER LIMIT

General Rule

$$\boxed{\binom{r}{k} = (-1)^k \binom{k-r-1}{k}} \qquad r \in R, \quad k \in Z$$

Formula is true even when $r \in R$

$$\boxed{\binom{x}{k} = (-1)^k \binom{k-x-1}{k}} \; ! \qquad x \in R, \quad k \in Z$$

Proof

$$x^{\underline{k}} = x(x-1)\dots(x-k+1)$$

$$= (-1)^k(-x)(1-x)(2-x)\dots(k-1-x)$$

$$= \boxed{(-1)^k(k-x-1)^{\underline{k}}}$$

Evaluate

$$(k-x-1)^{\underline{k}} = (k-x-1)(k-x-1-1)\dots$$

$$\underbrace{(k-x-1-k+1)}_{(-x)}$$

**REMARK** (negate twice!)

$$\binom{x}{k} \overset{①}{=} (-1)^k \binom{k-x-1}{k} =$$

$$\overset{①}{=} (-1)^k (-1)^k \binom{k-(k-x-1)-1}{k}$$

$$= (-1)^{2k}\binom{x}{k} = \binom{x}{k}$$

①

$$\boxed{\binom{x}{k} = (-1)^k \binom{k-x-1}{k}}$$

SYMMETRY

② $$\boxed{(-1)^m \binom{-n-1}{m} = (-1)^n \binom{-m-1}{n}}$$

Proof from ①

$$\text{LEFT} = (-1)^m \binom{-n-1}{m} = (-1)^m (-1)^m \binom{m-(-n-1)-1}{m}$$

$$= (-1)^{2m}\binom{m+n}{m} = \binom{m+n}{m}$$

$$\boxed{\text{LEFT} = \binom{m+n}{m}}$$

$m-(-n-1)-1$

$= m+n+1-1$

4)

$$\text{RIGHT} = (-1)^n \binom{\overbrace{-m-1}^{x}}{\underset{k}{n}}$$

$$\boxed{\binom{x}{k} = (-1)^k \binom{k-x-1}{k}} \quad (1)$$

$$\overset{(1)}{=} (-1)^n (-1)^n \binom{\overset{k}{n} - \overset{x}{(-m-1)} - 1}{n}$$

$$= (-1)^{2n} \binom{n+m}{n}$$

$$\boxed{\text{RIGHT} = \binom{n+m}{n}}$$

LEFT = RIGHT means that

$$\boxed{\binom{n+m}{m} = \binom{n+m}{n}}$$

for $n, m \in \mathbb{Z}$

Proof: use $\boxed{\binom{n}{k} = \binom{n}{n-k}}$

$$\binom{n+m}{m} = \binom{n+m}{n+m-m} = \binom{n+m}{n}$$

<u>yes</u>!

EVALUATE:

$$\sum_{k\le m}\binom{x}{k}(-1)^k = \binom{x}{0} - \binom{x}{1} + \dots + (-1)^m\binom{x}{m}$$

$k<0$ terms $=0$

SHOW

$$\boxed{\sum_{k\le m}\binom{x}{k}(-1)^k = (-1)^m\binom{x-1}{m}}$$

$m\in\mathbb{Z}$
$x\in\mathbb{R}$

$$\boxed{① \quad \binom{x}{k} = (-1)^k\binom{k-x-1}{k}}$$

LEFT:

$$\sum_{k\le m}\binom{x}{k}(-1)^k \overset{①}{=} \sum_{k\le m}(-1)^k(-1)^k\binom{k-x-1}{k}$$

use

$$\boxed{\sum_{k\le m}\binom{x+k}{k} = \binom{x+m+1}{m}}$$

$$= \sum_{k\le m}\binom{k-x-1}{k} \longrightarrow$$

$$= \sum_{k\le m}\binom{\overbrace{-x-1}^{x}+k}{k} = \binom{-x-1+m+1}{m}$$

$$= \binom{\overbrace{-x+m}^{x}}{m} \overset{①}{=} (-1)^m\binom{m-(-x+m)-1}{m} = (-1)^m\binom{x-1}{m}$$

$=$ RIGHT

We proved: ①

$$\boxed{\sum_{k \le m} \binom{x}{k}(-1)^k = (-1)^m \binom{x-1}{m}}$$

CLOSED formula

what about

$$\boxed{\sum_{k \le m} \binom{n}{k} = \binom{n}{0} + \binom{n}{1} + \cdots \binom{n}{m}}$$

There is no closed formula!

~~HERE~~ THERE is ONE:

③ $$\boxed{\sum_{k \le m} \binom{x}{k}\left(\frac{x}{2} - k\right) = \frac{m+1}{2}\binom{x}{m+1}}$$ $m \in \mathbb{Z}$

proof by induction over m

④ $$\boxed{\sum_{k \le m} \binom{m+r}{k} x^k y^{m-k} = \sum_{k \le m} \binom{-r}{k}(-x)^k (x+y)^{m-k}}$$

proof: induction p. 166 HOMEWORK.

(5) $$\boxed{\binom{r}{m}\binom{m}{k} = \binom{r}{k}\binom{r-k}{m-k}}$$ $m, k \in \mathbb{Z}$, $r \in \mathbb{R}$ !

or in our $x$-notation

(5) $$\boxed{\binom{x}{m}\binom{m}{k} = \binom{x}{k}\binom{x-k}{m-k}}$$ $\boxed{x \in \mathbb{R}}$, $m, k \in \mathbb{Z}$

case $k=1$

$$\binom{x}{m}\binom{m}{1} = \binom{x}{1}\binom{x-k}{m-k}$$

$$\binom{x}{1} = x$$

$$\binom{m}{1} = m$$

$$\boxed{m\binom{x}{m} = x\binom{x-k}{m-k}}$$ Already proved!

use of (5): in $\Sigma$

$$\sum_m \binom{x}{m}\binom{m}{k} = \sum_m \binom{x}{k}\binom{x-k}{m-k}$$

$$= \binom{x}{k}\sum_m \binom{x-k}{m-k}$$ simpler sum

(5) $\boxed{\binom{x}{m}\binom{m}{k} = \binom{x}{k}\binom{x-k}{m-k}}$ $\boxed{\begin{array}{c} x \in R \\ m, r \in Z \end{array}}$

PROOF

STEP 1: proof case (combinatorial)
$x, m, k \in N$

STEP 2: extend $m, k$ to $Z$

STEP 3: Extend STEP 2 to $x \in R$ by POLYNOMIAL ARGUMENT

ST1 Assume $m, k, x \in N$ i.e $m, k, r \in N_{m<k}$

$$\boxed{\binom{r}{m}\binom{m}{k}} = \frac{r!}{m!(r-m)!} \cdot \frac{m!}{k!(m-k)!}$$

$$= \frac{r!}{k!(r-m)!(m-k)!(r-k)!}(r-k)!$$

$$= \frac{r!}{k!(r-k)!} \cdot \frac{(r-k)!}{(m-k)!(r-m)!} = \boxed{\binom{r}{k}\binom{r-k}{m-k}}$$

$$r - m = (r-k) - (m-k)$$

STEP 2   $K<0,\ m<K,\ m<0$

both sides of (5) are $=0$

STEP 3 : $L(x) - R(x) = 0$ for all $x \in N$

Both polynomials of degree $m$

so $L(x) = R(x)$ for all $x \in R$(eal)

(6) VANDERMONDE CONVOLUTION

$$\boxed{\sum_{K \in Z} \binom{x}{m+K}\binom{s}{n-K} = \binom{x+s}{m+n}}$$

$x \in R$
$n, m \in Z$
$s \in Z$

History: combinatorial version known in 1700 by Vandermonde (paper)

in 1303 by Chu-Shih-Chieh (China)

Proof : COMBINATORIAL :

$\boxed{n, m \in N \quad s \in N,\ x \in N}$

STEP 2 extend to $Z$

STEP 3 polynomial arg

proof of

$$⑥ \quad \boxed{\sum_{k \in \mathbb{Z}} \binom{x}{m+k}\binom{s}{n-k} = \binom{x+s}{n+m}}$$

$x \in \mathbb{R}$

$n, m, s \in \mathbb{Z}$

$k \in \mathbb{Z}$

combinatorial: all in $\mathbb{N}$.

REPLACE in ⑥ $\boxed{\begin{array}{l} \cancel{k \to k-m} \\ \cancel{n \to n-m} \end{array}}$ for the induction $m \neq 0$

we get case $m=0$

$$\boxed{\sum_{k \in \mathbb{N}} \binom{r}{k}\binom{s}{n-k} = \binom{r+s}{n}}$$

$n \in \mathbb{Z}$

$r, s \in \mathbb{N}$

⊕ Induction over $m$

⊕ $k, r, m < 0$ argument (both sides $= 0$)

⊕ POLYNOMIAL argument

$\binom{r+s}{n}$ is # of ways to choose $n$ people among $r$ men + $s$ women

Each term of the $\Sigma$ is # ways to choose $k$ (people) out of $r$ men

$n-k$ people out of $s$ woman.

$n$

(7)
$$\boxed{\sum_{k\in\mathbb{Z}} \binom{l}{m+k}\binom{s}{n+k} = \binom{l+s}{l-m+n}} \quad l\in\mathbb{N},\ n,m\in\mathbb{Z}$$

Use: (6) + $\binom{n}{k} = \binom{n}{n-k}$ i.e. $\binom{l}{m+k} \overset{(*)}{=} \binom{l}{l-(m+k)}$

$$\sum_{k\in\mathbb{Z}} \binom{l}{m+k}\binom{s}{n+k} \overset{(*)}{=} \sum_{k\in\mathbb{Z}} \binom{l}{\underbrace{l-m}_{m} + \underbrace{(-k)}_{k}}\binom{s}{\underbrace{n-(-k)}_{n-k}}$$

$$\overset{(6)}{=} \binom{l+s}{l-m+n}$$ end.

(8)
$$\boxed{\sum_{k} \binom{l}{m+k}\binom{s+k}{n}(-1)^k = (-1)^{l+m}\binom{s-m}{n-l}} \quad l\in\mathbb{N},\ n\in\mathbb{Z},\ m\in\mathbb{Z}$$

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Proof: induction over $l\in\mathbb{N}$

use
$$\binom{l}{m+k} = \binom{l-1}{m+k} + \binom{l-1}{m+k-1}$$

for INDUCTIVE step.

+ for algebra others

(9)

$$\boxed{\sum_k \binom{m-r+s}{k}\binom{n+r-s}{n-k}\binom{r+k}{n+m} = \binom{r}{m}\binom{s}{n}}$$

$n, m \in \mathbb{N}$

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(10)

$$\boxed{\sum_k \binom{a+b}{a+k}\binom{b+c}{b+k}\binom{c+a}{c+k}(-1)^k = \frac{(a+b+c)!}{a!\,b!\,c!}}$$

(11)

$$\boxed{\sum_k \binom{a+b}{a+k}\binom{b+a}{b+k}(-1)^k = \frac{(a+b)!}{a!\,b!}}$$

(12)

John Dougall, 1900

$$\sum (-1)^k \binom{a+b}{a+k}\binom{b+c}{b+k}\binom{c+d}{c+k}\binom{d+a}{d+k} \Big/ \binom{2(a+b+c+d)}{a+b+c+d}$$

a,b,c,d $\in \mathbb{N}$

$$= \frac{(a+b+c+d)!\,(a+b+c)!\,(a+b+d)!\,(a+c+d)!\,(b+c+d)!}{(2(a+b+c+d))!\,(a+c)!\,(b+d)!\,a!\,b!\,c!\,d!}$$

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