CHAPTER 4 PROBLEM 45
The Number 9376 has a peculiar self-reproducing property that
\[(9376)^2 = 97909376\]

How many 4 Digit numbers \(x\) satisfy the equation
\[x^2 \mod 10000 = x?\]
[No Hints in Text Book 😞]
So the Problem can be restated as

\[ x^2 \mod 10^4 = x \]
\[ x^2 \equiv x \pmod{10^4} \]

[From textbook]

But we will prove it for general formula for \( n \) digits

\[ x^2 \equiv x \pmod{10^n} \]

\[ \text{(1)} \]
\( x \equiv x \pmod{10^n} \) \hspace{1cm} (2) \hspace{1cm} [\text{by Definition of Mod}] 

\[(1) - (2)\]
\[x^2 - x \equiv x - x \pmod{10^n}\]
\[x(x - 1) \equiv 0 \pmod{10^n}\]

we know that we can subtract congruence elements without losing congruence
Also, from
\[ x(x - 1) \equiv 0 \pmod{10^n}, \]
we have
\[ x(x - 1) \equiv 0 \pmod{2^n} \]
\[ x(x - 1) \equiv 0 \pmod{5^n} \]
[By Theorem of Independent Residues]
\[ x \mod 2^n = [0 \text{ or } 1] \]
[either \( x \text{ or } (x-1) \) have to be odd or even]

\[ x \mod 5^n = [0 \text{ or } 1] \]
[either \( x \text{ or } (x-1) \) has to be a multiple of 5, \( x \) has to be 5 or 6]
\[ x = 0 \mid x = 1 \mid x = 5 \mid x = 6 \]

First two hold good only when \( n = 1 \)

First Solution:
\[ x \equiv 1 \pmod{2^n} \]
\[ x \equiv 0 \pmod{5^n} \]

Second Solution:
\[ x \equiv 0 \pmod{2^n} \]
\[ x \equiv 1 \pmod{5^n} \]

Sum of the two Solutions is \( 10^n + 1 \) (from Wiki)
Thus the solutions are
\[ x \text{ and } 10^n + 1 - x \]

For \( n = 4 \)
\[ x \text{ and } 10^4 + 1 - x \]

we know \( x \) can be 9376
so the other number is
\[ 10000 + 1 - 9376 = 625 \]

But this is not a 4 digit number.
Thus for \( n = 4 \) there is only one 4 digit Automorphic Number.

But in general, for each \( n \), there are two \( n \) digit numbers [Not Proved]
References

www.Wikipedia.com
www.Mathworld.com