CSE 547
DISCRETE MATHEMATICS
chapter 3 problem 31
Chapter 3 – Problem 31

Prove or disprove:
\[ |x| + |y| + |x + y| \leq |2x| + |2y| \]

We will consider the following cases:

CASE 1: When x and y both are integers
CASE 2: When x is real and y is integer
CASE 3: When x is integer and y is real
CASE 4: When x and y both are real

To evaluate the above cases let us review the properties pertaining to floors.
Properties Review

• If \( x \) is an integer,
  \[ [x] = x \]  \hspace{1cm} \text{...property(1)}

• If \( x \) is real then \( x \) can be represented as:
  \[ x = [x] + \{x\} \]  \hspace{1cm} \text{...property(2)}

  where \( \{x\} \) is the fractional part of \( x \)

• If \( \{x\} < \frac{1}{2} \) \( \{2x\} = 2\{x\} \) and  \hspace{1cm} \text{...property(3)}

• If \( \{x\} > \frac{1}{2} \) \( \{2x\} = 2\{x\} - 1 \)  \hspace{1cm} \text{...property(4)}

• If \( x \) and \( y \) are real then,
  \[ \{x + y\} = \{\{x\} + \{y\}\} \]  \hspace{1cm} \text{...property(5)}

• If \( n < x < n+1 \) then,
  \[ [x] = n \]  \hspace{1cm} \text{...from (3.5-a)}

  \[ \therefore \{x\} = x - n \]  \hspace{1cm} \text{..from(2) ... property(6)}
Properties 3 & 4 proof...

If $2x$ is real then

$$2x = \lceil 2x \rceil + \{2x\} \quad \text{...equation(1)...from(2)}$$

If $x$ is real,

$$x = \lceil x \rceil + \{x\} \quad \text{...from(2)}$$

$$2x = 2(\lceil x \rceil + \{x\})$$

$$2x = 2\lceil x \rceil + 2\{x\} \quad \text{...equation(2)}$$

$$\therefore \lceil 2x \rceil + \{2x\} = 2\lceil x \rceil + 2\{x\} \quad \text{...from(1) & (2)}$$

Equating the fractional part on both the sides,

$$\{\lceil 2x \rceil + \{2x\}\} = \{2\lceil x \rceil + 2\{x\}\}$$

Since $\lceil x \rceil$ is an integer $2\lceil x \rceil$ is also an integer. Since $\lceil 2x \rceil$ and $2\lceil x \rceil$ are integers, they do not contribute to the fractional part.

Hence, $\{\{2x\}\} = \{2x\} = \{2\{x\}\} \quad \text{... equation(3)}$
If \( \{x\} < \frac{1}{2} \)

\[
2\{x\} < 1
\]

\[
\therefore \{2\{x\}\} = 2\{x\} \quad \text{...equation(4)}
\]

If \( \{x\} > \frac{1}{2} \)

\[
2\{x\} > 1
\]

\[
\exists \{2\{x\}\} = 2\{x\} - 1 \quad \text{...equation(5)... from(6)}
\]

From equations (3),(4) and (5)

- If \( \{x\}<1/2 \) \( \{2x\} = 2\{x\} \) and \( \text{...property(3)} \)
- If \( \{x\}>1/2 \) \( \{2x\} = 2\{x\} - 1 \) \( \text{...property(4)} \)

Proved.
Properties 5 proof...

To prove: \( \{x + y\} = \{\{x\} + \{y\}\} \) \ ...property(5)

If \( x \) and \( y \) are real,
\[ (x+y) = [x + y] + \{x+y\} \] \ ...equation(6)...from(2)

Also \( x = [x] + \{x\} \) \ ...equation(7)...from(2)

And \( y = [y] + \{y\} \) \ ...equation(8)...from(2)

Adding (7) & (8)
\[ x+y = [x] + \{x\} + [y] + \{y\} \] \ ...equation(9)

Equating the fractional part on both the sides,
\[ \{[x + y] + \{x+y\}\} = \{[x] + \{x\} + [y] + \{y\}\} \ ...from(6)&(9) \]

Since \([x], [y] \) and \([x+y]\) are integers, they do not contribute to the fractional part.

\[ \{\{x+y\}\} = \{x+y\} = \{\{x\}+\{y\}\} \]

Proved.
CASE 1: x and y both are integers

To prove: \([x] + [y] + [x + y] \leq [2x] + [2y]\)

**L.H.S:** \([x] + [y] + [x + y]\)
\[= x + y + x + y \quad \text{... by property (1)}\]
\[= 2x + 2y\]

**R.H.S:** \([2x] + [2y]\)
\[= 2x + 2y \quad \text{... by property (1)}\]

\[\therefore [x] + [y] + [x + y] \leq [2x] + [2y]\]
CASE 2: x is a real and y is an integer
To prove: \([x] + [y] + [x + y] \leq [2x] + [2y]\)

**L.H.S:**
\([x] + [y] + [x + y]\)
  
  \[
  = [x] + y + [x] + y \quad \text{... by property (1)}
  
  = 2[x] + 2y \quad \text{... by property (2)}
  
  = 2 \left( x - \{ x \} \right) + 2y \quad \text{... equation (10)}
  
  = 2x + 2y - 2 \{ x \}
  
**R.H.S:**
\([2x] + [2y]\)

  
  \[
  = [2x] + 2y \quad \text{... by property (1)}
  
  = (2x - \{ 2x \}) + 2y \quad \text{... by property (2)}
  
  = 2x + 2y - \{ 2x \} \quad \text{... equation (11)}
CASE 2: continued...

For \( \{x\} < 1/2 \)

LHS: \( 2x + 2y - 2\{x\} \) \hspace{1cm} \ldots \text{from (10)}

RHS: \( 2x + 2y - 2\{x\} \) \hspace{1cm} \ldots \text{from (11) & (3)}

\[ [x] + [y] + [x + y] = [2x] + [2y] \]

For \( \{x\} > 1/2 \)

LHS: \( 2x + 2y - 2\{x\} \) \hspace{1cm} \ldots \text{from (10)}

RHS: \( 2x + 2y - (2\{x\} - 1) \) \hspace{1cm} \ldots \text{from (11) & (4)}

\[ [x] + [y] + [x + y] < [2x] + [2y] \]

For \( 0 < \{x\} < 1 \)

\[ [x] + [y] + [x + y] \leq [2x] + [2y] \]
CASE 3: x is an integer and y is a real
To prove: \([x] + [y] + [x + y] \leq [2x] + [2y]\)

L.H.S: \([x] + [y] + [x + y]\)
\[= [x] + y + [x] + y \quad \cdots \text{by property (1)}\]
\[= 2x + 2 \lfloor y \rfloor \]
\[= 2x + 2( y - \{ y \} ) \quad \cdots \text{by property (2)}\]
\[= 2x + 2y - 2\{ y \} \quad \cdots \text{equation (12)}\]

R.H.S: \([2x] + [2y]\)
\[= [2x] + 2y \quad \cdots \text{by property (1)}\]
\[= 2x + (2y - \{2y\}) \quad \cdots \text{by property (2)}\]
\[= 2x + 2y - \{2y\} \quad \cdots \text{equation (13)}\]
CASE 3: continued...

For \( \{y\} < 1/2 \)
- **LHS**: \( 2x + 2y - 2\{y\} \) ... from (12)
- **RHS**: \( 2x + 2y - 2\{y\} \) ... from (13) & (3)

\[ [x] + [y] + [x + y] = [2x] + [2y] \]

For \( \{y\} > 1/2 \)
- **LHS**: \( 2x + 2y - 2\{y\} \) ... from (12)
- **RHS**: \( 2x + 2y - (2\{y\} - 1) \) ... from (13) & (4)

\[ [x] + [y] + [x + y] < [2x] + [2y] \]

For \( 0 < \{y\} < 1 \)

\[ [x] + [y] + [x + y] \leq [2x] + [2y] \]
Case 4: x and y both are real
To prove: \( |x| + |y| + |x + y| \leq |2x| + |2y| \)

L.H.S: \( |x| + |y| + |x + y| \)
\[= x - \{x\} + y - \{y\} + (x + y) - \{x + y\} \quad \text{by property (2)} \]
\[= 2x + 2y - (\{x\} + \{y\} + \{x + y\}) \]
\[= 2x + 2y - (\{x\} + \{y\} + \{\{x\} + \{y\}\}) \quad \text{equation (14)} \]
\[\quad \text{from (5)} \]

R.H.S: \( |2x| + |2y| \)
\[= 2x - \{2x\} + 2y - \{2y\} \quad \text{by property (2)} \]
\[= 2x + 2y - (\{2x\} + \{2y\}) \quad \text{equation (15)} \]
Case 4: x and y both are real

To prove: \([x] + [y] + [x + y] \leq [2x] + [2y]\)

- **Case a:** \([x] < 1/2\) and \([y] < 1/2\)
  \([x] + [y] < 1\) always

**LHS:**
\[
2x + 2y - (\{x\} + \{y\} + \{x\} + \{y\}) \quad \text{... from (14)}
\]
\[
2x + 2y - (\{x\} + \{y\} + \{x\} + \{y\})
\]
\[
2x + 2y - 2\{x\} - 2\{y\}
\]

**RHS:**
\[
2x + 2y - (\{2x\} + \{2y\}) \quad \text{... from (15)}
\]
\[
2x + 2y - 2\{x\} - 2\{y\} \quad \text{... from (3)}
\]

For \(0 < \{x\} < 1/2\) and \(0 < \{y\} < 1/2\)
\([x] + [y] + [x + y] = [2x] + [2y]\)
Case 4: x and y both are real

To prove: \(|x| + |y| + |x + y| \leq |2x| + |2y|\)

- Case b: \(\{x\} < 1/2 \) and \(\{y\} > 1/2\)
  If \(\{x\} + \{y\} < 1\)

**LHS:**
\[
2x + 2y - (\{x\} + \{y\}) - (\{x\} + \{y\})
\]
\[
2x + 2y - 2\{x\} - 2\{y\}
\]

**RHS:**
\[
2x + 2y - (\{2x\} + \{2y\})
\]
\[
2x + 2y - 2\{x\} - 2\{y\} + 1
\]

\[\therefore |x| + |y| + |x + y| < |2x| + |2y|\]
Case 4: x and y both are real
To prove: \([x] + [y] + [x + y] \leq [2x] + [2y]\)

- **Case b:** \(\{x\} < 1/2\) and \(\{y\} > 1/2\)

  If \(\{x\} + \{y\} > 1\) and \(\{x\} + \{y\} < 2\)

  **LHS:** \(2x + 2y - (\{x\} + \{y\} + \{\{x\} + \{y\}\})\) \(... from(14)\)

  \[2x + 2y - (\{x\} + \{y\} + \{x\} + \{y\} - 1)\] \(... from (6)\)

  \[2x + 2y - 2\{x\} - 2\{y\} + 1\]

  **RHS:** \(2x + 2y - (\{2x\} + \{2y\})\) \(... from(15)\)

  \[2x + 2y - 2\{x\} - 2\{y\} + 1\] \(... from (3) & (4)\)

  \[\therefore [x] + [y] + [x + y] \leq [2x] + [2y]\]

  For 0 < \(\{x\}\) < 1/2 and 0 < \(\{y\}\) < 1

  \([x] + [y] + [x + y] \leq [2x] + [2y]\)
Case 4: x and y both are real

To prove: \([x] + [y] + [x + y] \leq [2x] + [2y]\)

- **Case c**: \(\{x\} > 1/2\) and \(\{y\} < 1/2\)
  
  If \(\{x\} + \{y\} < 1\)
  
  \[
  \text{LHS} : \quad 2x + 2y - (\{x\} + \{y\} + \{x\} + \{y\}) \quad \text{... from(14)} \\
  2x + 2y - (\{x\} + \{y\} + \{x\} + \{y\}) \\
  2x + 2y - 2\{x\} - 2\{y\} \\
  \text{RHS} : \quad 2x + 2y - (\{2x\} + \{2y\}) \quad \text{... from(15)} \\
  2x + 2y - 2\{x\} + 1 - 2\{y\} \quad \text{... from(3) & (4)}
  \]

\[\therefore [x] + [y] + [x + y] < [2x] + [2y]\]
Case 4: x and y both are real

To prove: \(|x| + |y| + |x + y| \leq \lfloor 2x \rfloor + \lfloor 2y \rfloor\)

- **Case c:** \({x}\) > \(1/2\) and \({y}\) < \(1/2\)

  If \(\{x\} + \{y\} > 1\) and \(\{x\} + \{y\} < 2\)

  \[\text{LHS: } 2x + 2y - (\{x\} + \{y\} + \{\{x\} + \{y\}\}) \quad \text{... from(14)}\]
  \[= 2x + 2y - (\{x\} + \{y\} + \{x\} + \{y\} - 1) \quad \text{... from(6)}\]
  \[= 2x + 2y - 2\{x\} - 2\{y\} + 1\]

  \[\text{RHS: } 2x + 2y - (\{2x\} + \{2y\}) \quad \text{... from(15)}\]
  \[= 2x + 2y - 2\{x\} + 1 - 2\{y\} \quad \text{... from(3) & (4)}\]

  \[\therefore |x| + |y| + |x + y| = \lfloor 2x \rfloor + \lfloor 2y \rfloor\]

For \(0 < \{x\} < 1\) and \(0 < \{y\} < 1/2\)

\[|x| + |y| + |x + y| \leq \lfloor 2x \rfloor + \lfloor 2y \rfloor\]
Case 4: x and y both are real

To prove: \([x] + [y] + [x + y] \leq [2x] + [2y]\)

- **Case d:** \(\{x\} > 1/2\) and \(\{y\} > 1/2\)

  \(\{x\} + \{y\} > 1\) always and \(\{x\} + \{y\} < 2\)

  **LHS:** \(2x + 2y - (\{x\} + \{y\} + \{\{x\} + \{y\}\})\) \hspace{1cm} \text{... from (14)}

  \(2x + 2y - (\{x\} + \{y\} + \{x\} + \{y\} - 1)\) \hspace{1cm} \text{... from (6)}

  \(2x + 2y - 2\{x\} - 2\{y\} + 1\)

  **RHS:** \(2x + 2y - (\{2x\} + \{2y\})\) \hspace{1cm} \text{... from (15)}

  \(2x + 2y - 2\{x\} + 1 - 2\{y\} + 1\) \hspace{1cm} \text{... from (3) \& (4)}

  \(\therefore [x] + [y] + [x + y] < [2x] + [2y]\)

For \(0 < \{x\} < 1\) and \(0 < \{y\} < 1\)

\([x] + [y] + [x + y] \leq [2x] + [2y]\)
To prove: \([x] + [y] + [x + y] \leq [2x] + [2y]\)

For all the 4 cases:

- **CASE 1:** When \(x\) and \(y\) both are integers
- **CASE 2:** When \(x\) is real and \(y\) is integer
- **CASE 3:** When \(x\) is integer and \(y\) is real
- **CASE 4:** When \(x\) and \(y\) both are real

\([x] + [y] + [x + y] \leq [2x] + [2y]\)

Hence Proved.