# CSE 547
# DISCRETE MATHEMATICS
## chapter 3 problem 31

# Chapter 3 – Problem 31

Prove or disprove :

$\lfloor x \rfloor + \lfloor y \rfloor + \lfloor x + y \rfloor \leq \lfloor 2x \rfloor + \lfloor 2y \rfloor$

We will consider the following cases:

CASE 1: When x and y both are integers

CASE 2: When x is real and y is integer

CASE 3: When x is integer and y is real

CASE 4: When x and y both are real

To evaluate the above cases let us review the properties pertaining to floors.

# Properties Review

- If x is an integer,
  $\lfloor x \rfloor = x$ ...property(1)
- If x is real then x can be represented as:
  $x = \lfloor x \rfloor + \{x\}$ ...property(2)
  where $\{x\}$ is the fractional part of x
- If $\{x\}<1/2$ $\{2x\} = 2\{x\}$ and ...property(3)
- If $\{x\}>1/2$ $\{2x\} = 2\{x\} - 1$ ...property(4)
- If x and y are real then,
  $\{x + y\} = \{\{x\} + \{y\}\}$ ...property(5)
- If $n < x < n+1$ then,
  $\lfloor x \rfloor = n$ ...from (3.5-a)
  $\therefore \{x\} = x - n$ ..from(2) ... property(6)

# Properties 3 & 4 proof…

If $2x$ is real then

$2x = \lfloor 2x \rfloor + \{2x\}$ …equation(1)…from(2)

If $x$ is real,

$x = \lfloor x \rfloor + \{x\}$ …from(2)

$2x = 2(\lfloor x \rfloor + \{x\})$

$2x = 2\lfloor x \rfloor + 2\{x\}$ …equation(2)

$\therefore \lfloor 2x \rfloor + \{2x\} = 2\lfloor x \rfloor + 2\{x\}$ …from(1) & (2)

Equating the fractional part on both the sides,

$\{\lfloor 2x \rfloor + \{2x\}\} = \{2\lfloor x \rfloor + 2\{x\}\}$

Since $\lfloor x \rfloor$ is an integer $2\lfloor x \rfloor$ is also an integer. Since $\lfloor 2x \rfloor$ and $2\lfloor x \rfloor$ are integers, they do not contribute to the fractional part.

Hence, $\{\{2x\}\} = \{2x\} = \{2\{x\}\}$ … equation(3)

# Properties 3 & 4 proof continued…

If $\{x\} < 1/2$

$2\{x\} < 1$

$\therefore \{2\{x\}\} = 2\{x\}$ …equation(4)

If $\{x\} > 1/2$

$2\{x\} > 1$

$\therefore \{2\{x\}\} = 2\{x\} - 1$ …equation(5)… from(6)

From equations (3),(4) and (5)

- If $\{x\}<1/2 \quad \{2x\} = 2\{x\}$ and …property(3)
- If $\{x\}>1/2 \quad \{2x\} = 2\{x\} -1$ …property(4)

Proved.

# Properties 5 proof…

To prove : $\{x + y\} = \{\{x\} + \{y\}\}$ …property(5)

If x and y are real,

$(x+y) = \lfloor x + y \rfloor + \{x+y\}$ …equation(6)…from(2)

Also $x = \lfloor x \rfloor + \{x\}$ …equation(7)…from(2)

And $y = \lfloor y \rfloor + \{y\}$ …equation(8)…from(2)

Adding (7) & (8)

$x+y = \lfloor x \rfloor + \{x\} + \lfloor y \rfloor + \{y\}$ …equation(9)

Equating the fractional part on both the sides,

$\{\lfloor x + y \rfloor + \{x+y\}\} = \{\lfloor x \rfloor + \{x\} + \lfloor y \rfloor + \{y\}\}$ …from(6)&(9)

Since $\lfloor x \rfloor$ ,$\lfloor y \rfloor$ and $\lfloor x+y \rfloor$ are integers, they do not contribute to the fractional part.

$\{\{x+y\}\} = \{x+y\} = \{\{x\}+\{y\}\}$

Proved.

## CASE 1: x and y both are integers

## To prove : $\lfloor x \rfloor + \lfloor y \rfloor + \lfloor x + y \rfloor \leq \lfloor 2x \rfloor + \lfloor 2y \rfloor$

L.H.S : $\lfloor x \rfloor + \lfloor y \rfloor + \lfloor x + y \rfloor$

$= x + y + x + y$ ... by property (1)

$= 2x + 2y$

R.H.S : $\lfloor 2x \rfloor + \lfloor 2y \rfloor$

$= 2x + 2y$ ... by property (1)

$\therefore \lfloor x \rfloor + \lfloor y \rfloor + \lfloor x + y \rfloor \leq \lfloor 2x \rfloor + \lfloor 2y \rfloor$

# CASE 2: x is a real and y is an integer

## To prove : $\lfloor x \rfloor + \lfloor y \rfloor + \lfloor x + y \rfloor \leq \lfloor 2x \rfloor + \lfloor 2y \rfloor$

L.H.S : $\lfloor x \rfloor + \lfloor y \rfloor + \lfloor x + y \rfloor$

$= \lfloor x \rfloor + y + \lfloor x \rfloor + y$ ... by property (1)

$= 2 \lfloor x \rfloor + 2y$

$= 2 ( x - \{ x \} ) + 2y$ ... by property (2)

$= 2x + 2y - 2 \{ x \}$ ... equation(10)

R.H.S : $\lfloor 2x \rfloor + \lfloor 2y \rfloor$

$= \lfloor 2x \rfloor + 2y$ ... by property (1)

$= ( 2x - \{ 2x \} ) + 2y$ ... by property (2)

$= 2x + 2y - \{ 2x \}$ ... equation(11)

# CASE 2: continued…

For $\{x\} < 1/2$

LHS : $2x + 2y - 2\{x\}$ … from (10)

RHS : $2x + 2y - 2\{x\}$ … from (11) & (3)

$\therefore \lfloor x \rfloor + \lfloor y \rfloor + \lfloor x + y \rfloor = \lfloor 2x \rfloor + \lfloor 2y \rfloor$

For $\{x\} > 1/2$

LHS : $2x + 2y - 2\{x\}$ … from (10)

RHS : $2x + 2y - (2\{x\} - 1)$ … from (11) & (4)

⍰ $\lfloor x \rfloor + \lfloor y \rfloor + \lfloor x + y \rfloor < \lfloor 2x \rfloor + \lfloor 2y \rfloor$

For $0 < \{x\} < 1$

⍰ $\lfloor x \rfloor + \lfloor y \rfloor + \lfloor x + y \rfloor \leq \lfloor 2x \rfloor + \lfloor 2y \rfloor$

## CASE 3: x is an integer and y is a real

## To prove : $\lfloor x \rfloor + \lfloor y \rfloor + \lfloor x + y \rfloor \le \lfloor 2x \rfloor + \lfloor 2y \rfloor$

L.H.S : $\lfloor x \rfloor + \lfloor y \rfloor + \lfloor x + y \rfloor$

$= \lfloor x \rfloor + y + \lfloor x \rfloor + y$ ... by property (1)

$= 2x + 2 \lfloor y \rfloor$

$= 2x + 2( y - \{ y \} )$ ... by property (2)

$= 2x + 2y - 2\{ y \}$ ... equation(12)

R.H.S : $\lfloor 2x \rfloor + \lfloor 2y \rfloor$

$= \lfloor 2x \rfloor + 2y$ ... by property (1)

$= 2x + (2y - \{2y\})$ ... by property (2)

$= 2x + 2y - \{2y\}$ ... equation(13)

# CASE 3: continued…

For $\{y\} < 1/2$

LHS : $2x + 2y - 2\{y\}$ … from (12)

RHS : $2x + 2y - 2\{y\}$ … from (13) & (3)

$\therefore \lfloor x \rfloor + \lfloor y \rfloor + \lfloor x + y \rfloor = \lfloor 2x \rfloor + \lfloor 2y \rfloor$

For $\{y\} > 1/2$

LHS : $2x + 2y - 2\{y\}$ … from (12)

RHS : $2x + 2y - (2\{y\} - 1)$ … from (13) & (4)

⍰ $\lfloor x \rfloor + \lfloor y \rfloor + \lfloor x + y \rfloor < \lfloor 2x \rfloor + \lfloor 2y \rfloor$

For $0 < \{y\} < 1$

⍰ $\lfloor x \rfloor + \lfloor y \rfloor + \lfloor x + y \rfloor \leq \lfloor 2x \rfloor + \lfloor 2y \rfloor$

## Case 4: x and y both are real

## To prove : $\lfloor x\rfloor + \lfloor y\rfloor + \lfloor x + y\rfloor \leq \lfloor 2x\rfloor + \lfloor 2y\rfloor$

L.H.S: $\lfloor x\rfloor + \lfloor y\rfloor + \lfloor x + y\rfloor$

$= x - \{x\} + y - \{y\} + (x + y) - \{x + y\}$ ...by property (2)

$= 2x + 2y - (\{x\} + \{y\} + \{x + y\})$

$= 2x + 2y - (\{x\} + \{y\} + \{\{x\} + \{y\}\})$ ...equation(14)

...from (5)

R.H.S: $\lfloor 2x\rfloor + \lfloor 2y\rfloor$

$= 2x - \{2x\} + 2y - \{2y\}$ ...by property (2)

$= 2x + 2y - (\{2x\} + \{2y\})$ ...equation(15)

# Case 4: x and y both are real

## To prove : $\lfloor x \rfloor + \lfloor y \rfloor + \lfloor x + y \rfloor \leq \lfloor 2x \rfloor + \lfloor 2y \rfloor$

- *<u>Case a</u>*: $\{x\}<1/2$ and $\{y\}<1/2$

  $\{x\}+\{y\}<1$ always

LHS : $2x + 2y - ( \{ x \} + \{ y \} + \{\{x\} +\{y\}\} )$ ... from(14)

$2x + 2y - (\{x\}+\{y\}+\{x\}+\{y\})$

$2x + 2y - 2\{x\} -2\{y\}$

RHS : $2x + 2y - ( \{ 2x \} + \{ 2y \})$ ... from(15)

$2x + 2y - 2\{x\} - 2\{y\}$ ... from(3)

For $0 < \{x\} < 1/2$ and $0 < \{y\} < 1/2$

$\lfloor x \rfloor + \lfloor y \rfloor + \lfloor x + y \rfloor = \lfloor 2x \rfloor + \lfloor 2y \rfloor$

# Case 4: x and y both are real
## To prove : $\lfloor x \rfloor + \lfloor y \rfloor + \lfloor x + y \rfloor \leq \lfloor 2x \rfloor + \lfloor 2y \rfloor$

- *Case b*: $\{x\}<1/2$ and $\{y\}>1/2$

  If $\{x\}+\{y\} < 1$

LHS : $2x + 2y - ( \{ x \} + \{ y \} + \{\{x\} +\{y\}\} )$ ... from(14)

$2x + 2y - (\{x\}+\{y\}+\{x\}+\{y\})$

$2x + 2y - 2\{x\} -2\{y\}$

RHS : $2x + 2y - ( \{ 2x \} + \{ 2y \})$ ... from(15)

$2x + 2y - 2\{x\} - 2\{y\} + 1$ ... from(3) & (4)

$\therefore \lfloor x \rfloor + \lfloor y \rfloor + \lfloor x + y \rfloor < \lfloor 2x \rfloor + \lfloor 2y \rfloor$

# Case 4: x and y both are real

# To prove : $\lfloor x \rfloor + \lfloor y \rfloor + \lfloor x + y \rfloor \leq \lfloor 2x \rfloor + \lfloor 2y \rfloor$

- *Case b*: $\{x\}<1/2$ and $\{y\}>1/2$

  If $\{x\}+\{y\} > 1$ *and* $\{x\}+\{y\} < 2$

LHS : $2x + 2y - ( \{ x \} + \{ y \} + \{\{x\} +\{y\}\} )$ ... from (14)

$2x + 2y - (\{x\}+\{y\}+\{x\}+\{y\} - 1)$ ... from (6)

$2x + 2y - 2\{x\} -2\{y\} + 1$

RHS : $2x + 2y - ( \{ 2x \} + \{ 2y \})$ ... from (15)

$2x + 2y - 2\{x\} - 2\{y\} + 1$ ... from (3) & (4)

$\therefore \lfloor x \rfloor + \lfloor y \rfloor + \lfloor x + y \rfloor \leq \lfloor 2x \rfloor + \lfloor 2y \rfloor$

For $0 < \{x\} < 1/2$ and $0 < \{y\} < 1$

$\lfloor x \rfloor + \lfloor y \rfloor + \lfloor x + y \rfloor \leq \lfloor 2x \rfloor + \lfloor 2y \rfloor$

# Case 4: x and y both are real

## To prove : $\lfloor x \rfloor + \lfloor y \rfloor + \lfloor x + y \rfloor \leq \lfloor 2x \rfloor + \lfloor 2y \rfloor$

- *Case c*: $\{x\} > 1/2$ and $\{y\} < 1/2$

  If $\{x\} + \{y\} < 1$

LHS : $2x + 2y - ( \{ x \} + \{ y \} + \{\{x\} + \{y\}\} )$ ... from (14)

$2x + 2y - (\{x\} + \{y\} + \{x\} + \{y\})$

$2x + 2y - 2\{x\} - 2\{y\}$

RHS : $2x + 2y - ( \{ 2x \} + \{ 2y \})$ ... from (15)

$2x + 2y - 2\{x\} + 1 - 2\{y\}$ ... from (3) & (4)

$\therefore \lfloor x \rfloor + \lfloor y \rfloor + \lfloor x + y \rfloor < \lfloor 2x \rfloor + \lfloor 2y \rfloor$

# Case 4: x and y both are real

## To prove : $\lfloor x \rfloor + \lfloor y \rfloor + \lfloor x + y \rfloor \leq \lfloor 2x \rfloor + \lfloor 2y \rfloor$

- *<u>Case c</u>*: $\{x\}>1/2$ and $\{y\}<1/2$

  If *$\{x\}+\{y\} > 1$ and $\{x\}+\{y\} < 2$*

LHS : $2x + 2y – ( \{ x \} + \{ y \} + \{\{x\} +\{y\}\} )$ ... from(14)

$2x + 2y – (\{x\}+\{y\}+\{x\}+\{y\} - 1)$ ... from(6)

$2x + 2y – 2\{x\} -2\{y\} + 1$

RHS : $2x + 2y – ( \{ 2x \} + \{ 2y \})$ ... from(15)

$2x + 2y – 2\{x\} + 1- 2\{y\}$ ... from(3) & (4)

$\therefore \lfloor x \rfloor + \lfloor y \rfloor + \lfloor x + y \rfloor = \lfloor 2x \rfloor + \lfloor 2y \rfloor$

For $0 < \{x\} < 1$ and $0 < \{y\} < 1/2$

$\lfloor x \rfloor + \lfloor y \rfloor + \lfloor x + y \rfloor \leq \lfloor 2x \rfloor + \lfloor 2y \rfloor$

## Case 4: x and y both are real

## To prove : $\lfloor x \rfloor + \lfloor y \rfloor + \lfloor x + y \rfloor \leq \lfloor 2x \rfloor + \lfloor 2y \rfloor$

- *Case d*: $\{x\}>1/2$ and $\{y\}>1/2$

  *$\{x\}+\{y\} > 1$ always and $\{x\}+\{y\} < 2$*

LHS : $2x + 2y - ( \{ x \} + \{ y \} + \{\{x\} + \{y\}\} )$ ... from (14)

$2x + 2y - (\{x\}+\{y\}+\{x\}+\{y\} - 1)$ ... from (6)

$2x + 2y - 2\{x\} - 2\{y\} + 1$

RHS : $2x + 2y - ( \{ 2x \} + \{ 2y \})$ ... from (15)

$2x + 2y - 2\{x\} + 1 - 2\{y\} + 1$ ... from (3) & (4)

$\therefore \lfloor x \rfloor + \lfloor y \rfloor + \lfloor x + y \rfloor < \lfloor 2x \rfloor + \lfloor 2y \rfloor$

For $0 < \{x\} < 1$ and $0 < \{y\} < 1$

$\lfloor x \rfloor + \lfloor y \rfloor + \lfloor x + y \rfloor \leq \lfloor 2x \rfloor + \lfloor 2y \rfloor$

## To prove : $\lfloor x \rfloor + \lfloor y \rfloor + \lfloor x + y \rfloor \leq \lfloor 2x \rfloor + \lfloor 2y \rfloor$

For all the 4 cases :

- CASE 1: When x and y both are integers
- CASE 2: When x is real and y is integer
- CASE 3: When x is integer and y is real
- CASE 4: When x and y both are real

$$\lfloor x \rfloor + \lfloor y \rfloor + \lfloor x + y \rfloor \leq \lfloor 2x \rfloor + \lfloor 2y \rfloor$$

Hence Proved.