Problem Statement

Show that the $n$th element of the sequence: $1,2,2,3,3,3,4,4,4,4,5,5,5,5,5,\ldots$ is $\left\lfloor \sqrt{2n} + \frac{1}{2} \right\rfloor$.
Let the $n^{th}$ element be $m$. We need to find $m$

Let $P(x)$ represent the position of the last occurrence of $x$ in the sequence.

We show that $P(x) = \frac{x(x + 1)}{2}$ by induction:

Basis: It is true for $P(1)$ since $1(1+1)/2 = 1$

Since $x$ is $(x)$ places ahead of last occurrence of $x-1$, we have:

$$P(x) = P(x - 1) + x$$

$$= \frac{(x - 1)x}{2} + x$$

by inductive assumption

$$= \frac{(x - 1)x + 2x}{2}$$
Therefore \( P(x) = \frac{x(x + 1)}{2} \) for all \( x \in \mathbb{Z}^+ \)
Hence proved.
\[ P(m-1) = \frac{(m-1)m}{2} \]

\[ P(m) = \frac{m(m+1)}{2} \]

Since the \( n^{th} \) element should lie between \( P(m-1) \) and \( P(m) \), we have:

\[ \frac{(m-1)m}{2} < n \leq \frac{m(m+1)}{2} \]

\[ (m-1)m < 2n < m(m+1) < m(m+1) + 1/4 \]

obvious
\[ m^2 - m + \frac{1}{4} \leq 2n < m^2 + m + \frac{1}{4} \]

We add \( \frac{1}{4} \) - does not give next integer, but we can put \( \leq 2n \)

\[
\left( m - \frac{1}{2} \right)^2 \leq 2n < \left( m + \frac{1}{2} \right)^2
\]

\[
m - \frac{1}{2} \leq \sqrt{2n} < m + \frac{1}{2}
\]

Add \( \frac{1}{2} \) to all sides

\[
m = \left\lfloor \sqrt{2n} + \frac{1}{2} \right\rfloor
\]

Follows from of (3.5)(a) which is:

\[
[x] = n \iff n \leq x < n + 1
\]
Thank you