We extend the notion of a finite sum $\sum_{k=1}^{n} a_k$ to an INFINITE SUM which we write as

$$\sum_{n=1}^{\infty} a_n$$

as follows.
DEFINITION 1

For a given sequence 
\( \{a_n\}_{n \in \mathbb{N} - \{0\}} \), i.e the sequence 
\[ a_1, a_2, a_3, \ldots, a_n, \ldots \]

we form a following (infinite) sequence

\[ S_1 = a_1, S_2 = a_1 + a_2, \ldots, S_n = \sum_{k=1}^{n} a_k, \ldots \]

We use it to define the infinite sum as follows.
DEFINITION 1

If the limit of the sequence \( \{S_n\} \) exists we call it an INFINITE SUM of the sequence \( \sum_{k=1}^{n} a_k \).

We write it as

\[
\sum_{n=1}^{\infty} a_n = \lim_{n \to \infty} S_n = \lim_{n \to \infty} \sum_{k=1}^{n} a_k.
\]

The sequence \( \{S_n = \sum_{k=1}^{n} a_k\} \) is called its sequence of partial sums.
DEFINITION 2

If the limit \( \lim_{n \to \infty} S_n \) exists and is finite, i.e.

\[
\lim_{n \to \infty} S_n = S,
\]

then we say that the infinite sum

\[
\sum_{n=1}^{\infty} a_n \text{ CONVERGES to } S \text{ and }
\]

we write it as

\[
\sum_{n=1}^{\infty} a_n = S,
\]

otherwise the infinite sum DIVERGES.
In a case that
\[ \lim_{n \to \infty} S_n \]
exists and is infinite, i.e.
\[ \lim_{n \to \infty} S_n = \infty, \]

we say that the infinite sum
\[ \sum_{n=1}^{\infty} a_n \]
DIVERGES to \( \infty \) and

we write it as
\[ \sum_{n=1}^{\infty} a_n = \infty. \]

In a case that \( \lim_{n \to \infty} S_n \) does not exist we say that the infinite sum \( \sum_{n=1}^{\infty} a_n \) DIVERGES.
Observation 1 In a case when all elements of the sequence \( \{a_n\} \) are equal 0 starting from a certain \( k \geq 1 \) the infinite sum becomes a finite sum, hence the infinite sum is a generalization of the finite one, and this is why we keep the similar notation.

**EXAMPLE 1** The infinite sum of a geometric sequence \( a_n = x^k \) for \( x \geq 0 \), i.e.

\[
\sum_{n=1}^{\infty} x^n
\]

converges if and only if \( |x| < 1 \) because

\[
\sum_{k=1}^{n} x^k = S_n = \frac{x^{n+1} - x}{x - 1}, \quad \text{and}
\]

\[
\lim_{n \to \infty} S_n = \lim_{n \to \infty} \frac{x}{x - 1} (x^n - 1) = \frac{x}{x - 1} \quad \text{iff} \quad |x| < 1,
\]

hence

\[
\sum_{n=1}^{\infty} x^k = \frac{x}{x - 1}.
\]
EXAMPLE 2 The series $\Sigma_{n=1}^{\infty} 1$ DIVERGES to $\infty$ as $S_n = \Sigma_{k=1}^{n} 1 = n$ and

$$\lim_{n \to \infty} S_n = \lim_{n \to \infty} n = \infty.$$

EXAMPLE 3 The infinite sum $\Sigma_{n=1}^{\infty} (-1)^n$ DIVERGES.
**EXAMPLE 4** The infinite sum

\[
\sum_{n=1}^{\infty} \frac{1}{(k + 1)(k + 2)} \quad \text{CONVERGES and}
\]

\[
\sum_{n=1}^{\infty} \frac{1}{(k + 1)(k + 2)} = 1.
\]

**Proof:** first we evaluate \( S_n = \sum_{k=1}^{n} \frac{1}{(k + 1)(k + 2)} \) as follows.

\[
S_n = \sum_{k=1}^{n} \frac{1}{(k + 1)(k + 2)} = \sum_{k=1}^{n} k - 2
\]

\[
= -\frac{1}{x + 1}\bigg|_{0}^{n+1} = -\frac{1}{n + 2} + 1 \quad \text{and}
\]

\[
\lim_{n \to \infty} S_n = \lim_{n \to \infty} -\frac{1}{n + 2} + 1 = 1.
\]
DEFINITION 3 For any infinite sum (series) \( \sum_{n=1}^{\infty} a_n \) a series \( r_n = \sum_{m=n+1}^{\infty} a_m \) is called its n-th REMINDER.

FACT If \( \sum_{n=1}^{\infty} a_n \) converges, then so does its n-th REMINDER \( r_n = \sum_{m=n+1}^{\infty} a_m \).

Proof: first, observe that if \( \sum_{n=1}^{\infty} a_n \) converges, then for any value on \( n \) so does \( r_n = \sum_{m=n+1}^{\infty} a_m \) because

\[
r_n = \lim_{n \to \infty} (a_{n+1} + \ldots + a_{n+k}) = \lim_{n \to \infty} S_{n+k} - S_n = \sum_{m=1}^{\infty} a_m - S_n.
\]

So we get

\[
\lim_{n \to \infty} r_n = \sum_{m=1}^{\infty} a_m - \lim_{n \to \infty} S_n = \sum_{m=1}^{\infty} S_m - \sum_{n=1}^{\infty} a_n = S - S = 0.
\]
General Properties of Infinite Sums

THEOREM 1

If \( \sum_{n=1}^{\infty} a_n \) converges, then

\[
\lim_{n \to \infty} a_n = 0.
\]

Proof: observe that \( a_n = S_n - S_{n-1} \) and hence

\[
\lim_{n \to \infty} a_n = \lim_{n \to \infty} S_n The - \lim_{n \to \infty} S_{n-1} = 0,
\]

as \( \lim_{n \to \infty} S_n = \lim_{n \to \infty} S_{n-1} \).

REMARK The reverse statement to the theorem 1

If \( \lim_{n \to \infty} a_n = 0 \). then \( \sum_{n=1}^{\infty} a_n \) converges

is not always true. There are infinite sums with the term converging to zero that are not convergent.
EXAMPLE 5 The infinite HARMONIC sum

\[ H = \sum_{n=1}^{\infty} \frac{1}{n} \]

DIVERGES to \( \infty \), i.e.

\[ \sum_{n=1}^{\infty} \frac{1}{n} = \infty \]

but \( \lim_{n \to \infty} \frac{1}{n} = 0 \).
DEFINITION 4 Infinite sum

\[ \sum_{n=1}^{\infty} a_n \]

is BOUNDED if its sequence of partial sums

\[ S_n = \sum_{k=1}^{n} a_k \]

is BOUNDED; i.e. there is a number \( M \) such that

\[ |S_n| < M, \text{ for all } n \leq 1, n \in \mathbb{N}. \]

FACT 2 Every convergent infinite sum is bounded.
THEOREM 2 If the infinite sums
\[ \sum_{n=1}^{\infty} a_n, \quad \sum_{n=1}^{\infty} b_n \]
CONVERGE, then the following properties hold.

\[ \sum_{n=1}^{\infty} (a_n + b_n) = \sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} b_n, \]

\[ \sum_{n=1}^{\infty} c a_n = c \sum_{n=1}^{\infty} a_n, \quad c \in \mathbb{R}. \]
Alternating Infinite Sums

**DEFINITION 5** An infinite sum

\[ \sum_{n=1}^{\infty} (-1)^{n+1}a_n, \text{ for } a_n \geq 0 \]

is called ALTERNATING infinite sum (alternating series).

**EXAMPLE 6** Consider

\[ \sum_{n=1}^{\infty} (-1)^{n+1} = 1 - 1 + 1 - 1 + \ldots. \]

If we group the terms in pairs, we get

\[ (1 - 1) + (1 - 1) + \ldots = 0 \]

but if we start the pairing one step later, we get

\[ 1 - (1 - 1) - (1 - 1) - \ldots = 1 - 0 - 0 - 0 - \ldots = 1. \]
It shows that grouping terms in a case of infinite sum can lead to inconsistencies (contrary to the finite case). Look also example on page 59. We need to develop some strict criteria for manipulations and convergence/divergence of alternating series.
THEOREM 3 The alternating infinite sum
\[ \sum_{n=1}^{\infty} (-1)^{n+1}a_n, (a_n \geq 0) \]
such that
\[ a_1 \geq a_2 \geq a_3 \geq \ldots \text{ and } \lim_{n \to \infty} a_n = 0 \]
always CONVERGES.

Moreover, its partial sums
\[ S_n = \sum_{k=1}^{n} (-1)^{n+1}a_n \]
fulfil the condition
\[ S_{2n} \leq \sum_{n=1}^{\infty} (-1)^{n+1}a_n \leq S_{2n+1}. \]
**Proof:** observe that the sequence of $S_{2n}$ is increasing as

$$S'_{2n+2} = S_{2n} + (a_{2n+1} - a_{2n+2}$$

and

$$a_{2n+1} - a_{2n+2} \geq 0,$$

i.e.

$$S'_{2n+2} \geq S_{2n}.$$

The sequence of $S_{2n}$ is also bounded as

$$S_{2n} = a_1 - ((a_2 - a_3) + (a_4 - a_5) + ... a_{2n}) \leq a_1.$$

We know that any bounded and increasing sequence is is convergent, so we proved that $S_{2n}$ converges.
Let denote \( \lim_{n \to \infty} S_{2n} = g \).

To prove that
\[
\sum_{n=1}^{\infty} (-1)^{n+1} a_n = \lim_{n \to \infty} S_n
\]
converges we have to show now that
\[
\lim_{n \to \infty} S_{2n+1} = g.
\]

Observe that
\[
S_{2n+1} = S_{2n} + a_{2n+2}
\]
and we get
\[
\lim_{n \to \infty} S_{2n+1} = \lim_{n \to \infty} S_{2n} + \lim_{n \to \infty} a_{2n+2} = g
\]
as we assumed that
\[
\lim_{n \to \infty} a_n = 0.
\]

We proved that the sequence \( \{S_{2n}\} \) is increasing.
We prove in a similar way that the sequence \( \{S_{2n+1}\} \) is decreasing.

Hence we get

\[
S_{2n} \leq \lim_{n \to \infty} S_{2n} = g = \sum_{n=1}^{\infty} (-1)^{n+1}a_n
\]

and

\[
S_{2n+1} \geq \lim_{n \to \infty} S_{2n+1} = g
\]

and

\[
\sum_{n=1}^{\infty} (-1)^{n+1}a_n = g,
\]

i.e

\[
S_{2n} \leq \sum_{n=1}^{\infty} (-1)^{n+1}a_n \leq S_{2n+1}.
\]
EXAMPLE 7

Consider the ANHARMONIC series

\[ AH = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n} = \]

\[ 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} \ldots. \]

Observe that \( a_n = \frac{1}{n} \), and

\[ \frac{1}{n} \geq \frac{1}{n+1} \]

i.e. \( a_n \geq a_{n+1} \) for all \( n \).

This proves that the assumptions of the theorem 3 are fulfilled for \( AH \) and hence \( AH \) converges.

In fact, it is proved (by analytical methods) that

\[ AH = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n} = \ln 2. \]
EXAMPLE 8 A series (infinite sum)

\[ \sum_{n=0}^{\infty} (-1)^n \frac{1}{2n+1} \]

\[ = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} \ldots \]

CONVERGES, by Theorem 3.

Proof is similar to the one in the example 7).

It also is proved that

\[ \sum_{n=0}^{\infty} (-1)^n \frac{1}{2n+1} = \frac{\pi}{4}. \]
**THEOREM 4** (ABEL Theorem)

**IF** a sequence \( \{a_n\} \) fulfils the assumptions of the theorem 3 i.e.

\[
a_1 \geq a_2 \geq a_3 \geq \ldots \text{ and } \lim_{n \to \infty} a_n = 0
\]

and an infinite sum (converging or diverging)

\[
\sum_{n=1}^{\infty} b_n \text{ is bounded},
\]

**THEN** the infinite sum

\[
\sum_{n=1}^{\infty} a_n b_n
\]

always converges.

**Observe** that Theorem 3 is a special case of theorem 4 when \( b_n = (-1)^{n+1} \).
Convergence of Infinite Sums with Positive Terms

We consider now infinite sums with all its terms being positive real numbers, i.e.

\[ S = \sum_{n=1}^{\infty} a_n, \]

for \( a_n \geq 0, a_n \in \mathbb{R}. \)

Observe that if all \( a_n \geq 0, \) then the sequence \( \{S_n\} \) of partial sums is increasing; i.e.

\[ S_1 \leq S_2 \leq \ldots \leq S_n \ldots \]

and hence the limit

\[ \lim_{n \to \infty} S_n \]

exists and is finite or is \( \infty. \) This proves the following theorem.
THEOREM 5

The infinite sum

\[ S = \sum_{n=1}^{\infty} a_n, \text{ where } a_n \geq 0, a_n \in \mathbb{R} \]

always CONVERGES, or DIVERGES to \( \infty \).

THEOREM 6 (Comparing the series)

Let \( \sum_{n=1}^{\infty} a_n \) be an infinite sum and \( \{b_n\} \) be a sequence such that for all \( n \in \mathbb{N} \)

\[ 0 \leq b_n \leq a_n. \]

If the infinite sum \( \sum_{n=1}^{\infty} a_n \) converges then the sum \( \sum_{n=1}^{\infty} b_n \) also converges and

\[ \sum_{n=1}^{\infty} b_n \leq \sum_{n=1}^{\infty} a_n. \]
Proof: we denote

\[ S_n = \sum_{k=1}^{n} a_k, \quad T_n = \sum_{k=1}^{n} b_k. \]

As \( 0 \leq b_n \leq a_n \) we get that also

\[ S_n \leq T_n. \]

But

\[ S_n \leq \lim_{n \to \infty} S_n = \sum_{n=1}^{\infty} a_n \]

so also

\[ T_n \leq \sum_{n=1}^{\infty} a_n = S. \]
The inequality

\[ T_n \leq \sum_{n=1}^{\infty} a_n = S \]

means that the sequence \( \{T_n\} \) is a bounded sequence with positive terms,

hence by theorem 5, it converges.

By the assumption that

\[ \sum_{n=1}^{\infty} a_n \]

we get that

\[ \sum_{n=1}^{\infty} a_n = \lim_{n \to \infty} \sum_{k=1}^{n} a_k = \lim_{n \to \infty} S_n = S. \]
We just proved that

\[ T_n = \sum_{k=1}^{n} b_k \]

converges, i.e.

\[ \lim_{n \to \infty} T_n = T = \sum_{n=1}^{\infty} b_n. \]

But also we proved that

\[ S_n \leq T_n, \]

hence

\[ \lim_{n \to \infty} S_n \leq \lim_{n \to \infty} T_n \]

what means that

\[ \sum_{n=1}^{\infty} b_n \leq \sum_{n=1}^{\infty} a_n. \]
EXAMPLE 9

Let’s use Theorem 5 to prove that the series

\[ \sum_{n=1}^{\infty} \frac{1}{(n + 1)^2} \]

converges.

We prove by analytical methods that it converges to \( \frac{\pi^2}{6} \).

Here we prove only that it does converge.

First observe that the series below converges to 1, i.e.

\[ \sum_{n=1}^{\infty} \frac{1}{n(n + 1)} = 1. \]
Consider

\[ S_n = \frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \ldots + \frac{1}{n(n + 1)} = \]

\[ (1 - \frac{1}{2}) + (\frac{1}{2} - \frac{1}{3}) + \ldots (\frac{1}{n} - \frac{1}{n + 1}) = \]

\[ 1 - \frac{1}{n + 1} \]

so we get

\[ \sum_{n=1}^{\infty} \frac{1}{n(n + 1)} = \lim_{n \to \infty} S_n = \]

\[ \lim_{n \to \infty} (1 - \frac{1}{n + 1}) = 1. \]
Now we observe (easy to prove) that
\[
\frac{1}{2^2} \leq \frac{1}{1 \cdot 2}, \quad \frac{1}{3^2} \leq \frac{1}{1 \cdot 3}, \quad \ldots.
\]
\[
\frac{1}{(n + 1)^2} \leq \frac{1}{n(n + 1)}, \quad \ldots
\]
i.e. we proved that all assumptions if Theorem 5 hold, hence
\[
\sum_{n=1}^{\infty} \frac{1}{(n + 1)^2}
\]
converges and moreover
\[
\sum_{n=1}^{\infty} \frac{1}{(n + 1)^2} \leq \sum_{n=1}^{\infty} \frac{1}{n(n + 1)}.
\]
**THEOREM 7** (D’Alambert’s Criterium)

Any series with all its terms being positive real numbers, i.e.

\[ \sum_{n=1}^{\infty} a_n, \text{ for } a_n \geq 0, a_n \in \mathbb{R} \]

converges if the following condition holds:

\[ \lim_{n \to \infty} \frac{a_n}{a_{n+1}} < 1. \]

**Proof:** let \( h \) be any number such that

\[ \lim_{n \to \infty} \frac{a_n}{a_{n+1}} < h < 1. \]

It means that there is \( k \) such that for any \( n \geq k \) we have

\[ \frac{a_n}{a_{n+1}} < h \text{ and } a_{n+1} < ha_n. \]
Hence

\[ a_{k+1} < a_k h, \quad a_{k+2} = a_{k+1} h < a_k h^2, \ldots \]

i.e. all terms \( a_n \) of

\[ \sum_{n=k}^{\infty} a_n \]

are smaller than the terms of a converging (as \( 0 < h < 1 \)) geometric series

\[ \sum_{n=0}^{\infty} a_k h^n = a_k + a_k h + a_k h^2 + \ldots \]

By Theorem 5 the series

\[ \sum_{n=1}^{\infty} a_n \]

must converge.
THEOREM 7 (Cauchy’s Criterium)

Any series with all its terms being positive real numbers, i.e.

$$\sum_{n=1}^{\infty} a_n, \text{ for } a_n \geq 0, a_n \in \mathbb{R}$$

CONVERGES if the following condition holds:

$$\lim_{n \to \infty} \sqrt[n]{a_n} < 1.$$ 

Proof: we carry the proof in a similar way as the proof of theorem 6.
Let $h$ be any number such that

$$\lim_{n \to \infty} \sqrt[n]{a_n} < h < 1.$$ 

It means that there is $k$ such that for any $n \geq k$ we have $\sqrt[n]{a_n} < h$, i.e. $a_n < h^n$.

This means that all terms $a_n$ of $\sum_{n=k}^{\infty} a_n$ are smaller than the terms of a converging (as $0 < h < 1$) geometric series

$$\sum_{n=k}^{\infty} h^n = h^k + h^{k+1} + h^{k+2} + ...$$

By Theorem 5 the series $\sum_{n=1}^{\infty} a_n$ must converge.
THEOREM 7 (Divergence Criteria)

Any series with all its terms being positive real numbers, i.e.

$$\sum_{n=1}^{\infty} a_n, \text{ for } a_n \geq 0, a_n \in \mathbb{R}$$

DIVERGES if

$$\lim_{n \to \infty} \frac{a_n}{a_{n+1}} > 1$$

or

$$\lim_{n \to \infty} \sqrt[n]{a_n} > 1$$

Proof:

observe that if

$$\lim_{n \to \infty} \frac{a_n}{a_{n+1}} > 1,$$
then for sufficiently large $n$ we have that

\[
\frac{a_n}{a_{n+1}} > 1, \text{ and hence } a_{n+1} > a_n.
\]

This means that the limit of the sequence $\{a_n\}$ can’t be 0.

By theorem 1 we get that $\sum_{n=1}^{\infty} a_n$ diverges.
Similarly, if

$$\lim_{n \to \infty} \sqrt[n]{a_n} > 1,$$

then for sufficiently large $n$ we have that

$$\sqrt[n]{a_n} > 1 \text{ and hence } a_n > 1,$$

what means that the limit of the sequence $\{a_n\}$ can't be 0.

By theorem 1 we get that $\sum_{n=1}^{\infty} a_n$ diverges.
Remark It can happen that for a certain infinite sum $\sum_{n=1}^{\infty} a_n$

$$\lim_{n \to \infty} \frac{a_n}{a_{n+1}} = 1 = \lim_{n \to \infty} \sqrt[n]{a_n}.$$

In this case our Divergence Criteria do not decide whether the infinite sum converges or diverges.

In this case we say that the infinite sum **DOES NOT React** on the criteria.
EXAMPLE 10

The Harmonic series

\[ H = \sum_{n=1}^{\infty} \frac{1}{n} \]

does not react on D’Alambert’s Criterium (Theorem 7) because

\[ \lim_{n \to \infty} \frac{n}{n + 1} = \lim_{n \to \infty} \frac{1}{(1 + \frac{1}{n})} = 1. \]
EXAMPLE 11

The series from example 9

\[ \sum_{n=1}^{\infty} \frac{1}{(n + 1)^2} \]

does not react on D’Alambert’s Criterium (Theorem 7) because

\[ \lim_{n \to \infty} \frac{(n + 1)^2}{(n + 2)^2} = \lim_{n \to \infty} \frac{n^2 + 2n + 1}{n^2 + 4n + 1} = \]

\[ \lim_{n \to \infty} \frac{1 + \frac{2}{n} + \frac{1}{n^2}}{1 + \frac{4}{n} + \frac{4}{n^2}} = 1. \]
Remark

Both series

\[ \sum_{n=1}^{\infty} \frac{1}{n} \]

and

\[ \sum_{n=1}^{\infty} \frac{1}{(n + 1)^2} \]

**do not react on** D’Alambert’s, but first in divergent and the second is convergent.

There are more criteria for convergence, most known are Kumer’s criterium and Raabe criterium.