CHAPTER 2

INFINITE SUMS (SERIES)

Lecture Notes PART 2

1 Examples and Exercises

We consider now some examples and exercises.

EXAMPLE 1 Prove that
$$\Sigma_{n=1}^{\infty} \frac{c^n}{n!}$$

CONVERGES for $c > 0$.

Hint: use d'Alambert Criterium.

Proof: first we evaluate
$$\frac{a_{n+1}}{a_n} = \frac{c^{n+1}}{c^n} \cdot \frac{n!}{(n+1)!} = \frac{c}{n+1}.$$

Next we evaluate the limit
$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{c}{n+1} = 0 < 1.$$

By d'Alambert Criterium $\Sigma_{n=1}^{\infty} \frac{c^n}{n!}$ converges for $c > 0$. For $c < 0$ we get alternating series.

EXERCISE 2 Prove that the sequence $a_n = n!$ grows faster then the sequence $b_n = c^n$ for any $c > 0$.

Proof: we prove it by showing that
$$\lim_{n\to\infty} \frac{c^n}{n!} = 0.$$

Observe that we just proved that $\Sigma_{n=1}^{\infty} \frac{c^n}{n!}$ for any $c > 0$. By **Theorem 1** we get that $\lim_{n\to\infty} \frac{c^n}{n!} = 0$.

EXERCISE 3 Prove that the sequence $b_n = n^n$ grows faster then the sequence $a_n = n!$ for any $c > 0$.

Proof: we prove it, as before by showing that

$$\lim_{n\to\infty} \frac{n!}{n^n} = 0.$$

Observe that this is equivalent, by **Theorem 1** to proving convergence of the series $\Sigma_{n=1}^{\infty} \frac{n!}{n^n}$. We prove it as the following example.

EXAMPLE 2 Use d'Alambert Criterium to prove convergence of the following series:

$$\Sigma_{n=1}^{\infty} \frac{n!}{n^n}.$$

Proof: we evaluate

$$a_n = \frac{n!}{n^n}, \quad a_{n+1} = \frac{n!(n+1)}{(n+1)^n \cdot (n+1)}$$

and hence

$$\frac{a_{n+1}}{a_n} = \frac{n!(n+1)}{(n+1)^n(n+1)} = \frac{n^n}{(n+1)^n} = (\frac{n}{n+1})^n = \frac{1}{(1+\frac{1}{n})^n}.$$

Now we evaluate

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{1}{(1+\frac{1}{n})^n} = \frac{1}{e} < 1.$$

So by the d'Alambert Criterium $\Sigma_{n=1}^{\infty} \frac{n!}{n^n}$ converges.

EXAMPLE 3 The Harmonic series

$$H = \Sigma_{n=1}^{\infty} \frac{1}{n}$$

does not react on d'Alambert Criterium .

Proof: consider

$$\frac{a_{n+1}}{a_n} = \frac{1}{n+1} \cdot \frac{n}{1} = \frac{n}{n+1} = \frac{1}{1+\frac{1}{n}}$$

and

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{1}{1+\frac{1}{n}} = 1.$$

EXAMPLE 4

$$\lim_{n\to\infty} \frac{c^n}{n!} = 0, \text{ for } c > 0, \quad \lim_{n\to\infty} \frac{n!}{n^n} = 0.$$

Proof: follows directly from examples 1, 2 and **Theorem 1** that says:

If $\Sigma_{n=1}^{\infty} a_n$ converges, then $\lim_{n\to\infty} a_n = 0$.

EXAMPLE 5 We know that the Harmonic series $\Sigma_{n=1}^{\infty} \frac{1}{n}$ **diverges**. Use this information and **Cauchy Criterium** to prove that

$$\lim_{n \to \infty} \sqrt[n]{n} = 1.$$

Proof: observe that the sequence $a_n = \sqrt[n]{n}$ is for large n, decreasing and $a_n > 1$, hence $\lim_{n\to\infty} a_n$ exists and $\lim_{n\to\infty} \sqrt[n]{n} \geq 1$. Assume now that $\lim_{n\to\infty} \sqrt[n]{n} \neq 1$, i.e. that $\sqrt[n]{n} > 1$. This means that $\lim_{n\to\infty} \sqrt[n]{\frac{1}{n}} < 1$. That would prove, by **Cauchy Criterium** that $\Sigma_{n=1}^{\infty} \frac{1}{n}$ **converges** and we get a contradiction.

EXAMPLE 6 The series

$$\Sigma_{n=1}^{\infty} = \frac{|x(x-1)....(x-n+1)|}{n!} \cdot c^n$$

converges for $0 < c < 1$.

Proof: we evaluate

$$\frac{a_{n+1}}{a_n} = \frac{|x(x-1)....(x-n)|c^n c}{n!(n+1)} \cdot \frac{n!}{|x(x-1)....(x-n+1)|c^n} = \frac{|x-n|}{n+1} \cdot c,$$

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{|\frac{x}{n} - 1|}{1 + \frac{1}{n}} \cdot c = c.$$

By d'Alambert Criterium series converges for $0 < c < 1$.

EXAMPLE 7

$$\lim_{n\to\infty} \frac{|x(x-1)....(x-n+1)|}{n!} \cdot c^n \text{ for } 0 < c < 1.$$

Proof: Observe that this is equivalent, by **Theorem 1** to proving convergence of the series from **Example 6**, proved to be convergent.

2 Absolute and Conditional Convergence

We define the notions of absolute and conditional convergence as follows.

Definition of **absolute convergence.** The series

$$\Sigma_{n=1}^{\infty} a_n$$

converges **absolutly** if and only if the series

$$\Sigma_{n=1}^{\infty} |a_n|$$

converges.

Definition of **conditional convergence.** The series

$$\Sigma_{n=1}^{\infty} a_n$$

converges **conditionally** if and only if the series

$$\Sigma_{n=1}^{\infty} |a_n|$$

converges, but not absolutely, i.e. when

$$\Sigma_{n=1}^{\infty} a_n \text{ converges and } \Sigma_{n=1}^{\infty} |a_n| \text{does not converge.}$$

We state without the proof the following main theorem about absolute convergence.

Theorem 10 If the series $\Sigma_{n=1}^{\infty} a_n$ converges absolutely , then it converges. Moreover,

$$|\Sigma_{n=1}^{\infty} a_n| \leq \Sigma_{n=1}^{\infty} |a_n|.$$

Example 8 Geometric series $\Sigma_{n=1}^{\infty} aq^n$, $|q| < 1$ converges because the series $\Sigma_{n=1}^{\infty} |q|^n$ converges and $\Sigma_{n=1}^{\infty} |aq^n| = |a| \Sigma_{n=1}^{\infty} |q|^n$.

Example 9 The series

$$|\Sigma_{n=1}^{\infty} \frac{x^n}{n!}$$

converges **abolutely** for all $x \in R$. Moreover,

$$|\Sigma_{n=1}^{\infty} \frac{x^n}{n!} = e^x.$$

Proof: we proved, in Example 1 that it converges for $c > 0$, i.e. for $|x|$. The convergence to e^x is proved by other, analytical methods.

Example 10 The **enharmonic** series

$$\Sigma_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n}$$

converges **conditionally**.

Proof: we have

$$|a_n| = |(-1)^{n+1} \frac{1}{n}| = \frac{1}{n}$$

and the series $\Sigma_{n=1}^{\infty} |a_n| = \Sigma_{n=1}^{\infty} \frac{1}{n}$ diverges.

3 Finite and Infinite Commutativity

We know that the finite summation is commutative, i.e. that

$$\Sigma_{k=1}^{n} a_k = \Sigma_{k=1}^{n} a_{ik}$$

where a_{ik} is any **permutation** of $a_1, a_2, ...a_n$.

The commutativity **fails** for some infinite sums, as we have showed in Example 6 evaluating

$$\Sigma_{n=1}^{\infty}(-1)^{n+1} = 1 - 1 + 1 - 1 + ...$$

in two different ways (permutations).

If we group the terms in pairs, we get

$$(1 - 1) + (1 - 1) + ... = 0$$

but if we start the pairing one step later, we get

$$1 - (1 - 1) - (1 - 1) - ..... = 1 - 0 - 0 - 0 - ... = 1.$$

There are more examples in our book- pages 58-59.

QUESTION: when, for which class (if any) of infinite sums commutativity holds. Which are the classes (if any) of infinite sums commutativity fails. We have two basic Theorems (no proofs here).

Theorem 11 Every **absolutely convergent** infinite sum is **commutative**, i.e.

$$\Sigma_{n=1}^{\infty} a_n = \Sigma_{n=1}^{\infty} a_{mn}$$

for any permutation $m1, m2, ...mn....$ of natural numbers ≥ 1.

Theorem 11 is not true foe any convergent infinite sum; we can get two permutations build out of factors of enharmonic series $\Sigma_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n}$ in such way that one converges and other diverges to ∞.

Theorem 12 RIEMANN (1826-1866) **Theorem**

Any conditionally convergent infinite sum can be transformed by permutation of its factors into a sum that **diverges**, or to a sum that **converges** to any limit (finite or infinite).