CSE547
Chapter2 Problem 29

It is a short solution

Complete solutions is in the other file

Evaluate: $\sum_{k=1}^{n} \frac{(-1)^k k}{(4k^2 - 1)}$

$$\frac{k}{4k^2 - 1} = \frac{k}{(2k)^2 - (1)^2}$$

$$= \frac{k}{(2k - 1)(2k + 1)}$$

Using partial fractions,

$$\frac{k}{(2k - 1)(2k + 1)} = \frac{A}{(2k-1)} + \frac{B}{(2k+1)} \quad (\text{ eqn I })$$

Evaluating $\sum_{k=1}^{n} \frac{(-1)^k k}{(4k^2 - 1)}$

Multiplying and dividing by (2k -1) * (2k +1)

we get

$$k = A (2k + 1) + B (2k -1)$$

Grouping powers of k,

$$k + 0 = ((2A) k) + A + ((2B) k) – B$$

Equating powers of k on both sides, we get the linear equations:

$2A + 2B = 1$ and (equation 1)

$A - B = 0$ (equation 2)

Evaluating $\sum_{k=1}^{n} \frac{(-1)^k k}{(4k^2 - 1)}$

Solving the simultaneous equations obtained in the previous slide , we get

2 A + 2 B = 1

2A - 2 B = 0 (multiplying equation 2 by 2)

4A = 1 $\Rightarrow$ A = 1/4

From equation 2, B = 1/4

Evaluating $\sum_{k=1}^{n} \frac{(-1)^k k}{(4k^2 - 1)}$

$$\therefore \qquad \frac{k}{4k^2 - 1} = \frac{1}{4}\left(\frac{1}{(2k-1)} + \frac{1}{(2k+1)} \right)$$

$$\sum_{k=1}^{n} \frac{(-1)^k k}{(4k^2-1)} = \sum_{k=1}^{n} (-1)^k \frac{1}{4}\left(\frac{1}{(2k-1)} + \frac{1}{(2k+1)} \right)$$

Evaluating $\sum_{k=1}^{n} \frac{(-1)^k k}{(4k^2 - 1)}$

We can split the sum on the right side into two summations as follows :

$$\sum_{k=1}^{n} \frac{(-1)^k}{4} 1 \left(\frac{1}{(2k-1)} \right) + \sum_{k=1}^{n} \frac{(-1)^k}{4} 1 \left(\frac{1}{(2k+1)} \right)$$

This can be changed to a harmonic sum by putting $2k-1 = m$ and $2k+1 = m$ but that would make it complex.

Evaluating $\sum_{k=1}^{n} \frac{(-1)^k k}{(4k^2 - 1)}$

Expanding the summation we get

$$\frac{(-1)^1 1}{4}\left(\frac{1}{1} + \frac{1}{3}\right) + \frac{(-1)^2 1}{4}\left(\frac{1}{3} + \frac{1}{5}\right)$$

$$+ \quad \dots$$

$$+ \quad \frac{(-1)^n 1}{4}\left(\frac{1}{(2n - 1)} + \frac{1}{(2n + 1)}\right)$$

Evaluating $\sum_{k=1}^{n} \frac{(-1)^k k}{(4k^2 - 1)}$

Expanding the summation we get

$$\frac{1}{4}\left(\frac{(-1)}{1} + \frac{(-\cancel{1})}{\cancel{3}}\right) + \frac{1}{4}\left(\frac{\cancel{1}}{\cancel{3}} + \frac{\cancel{1}}{\cancel{5}}\right)$$

$$+ \quad \frac{1}{4}\left(\frac{(-\cancel{1})}{\cancel{5}} + \frac{(-\cancel{1})}{\cancel{7}}\right) + \quad \cdots \quad +$$

$$\frac{(-1)^n 1}{4}\left(\frac{\cancel{1}}{\cancel{(2n - 1)}} + \frac{1}{(2n + 1)}\right)$$

Evaluating $\sum_{k=1}^{n} \frac{(-1)^k k}{(4k^2 - 1)}$

We can see that the alternate terms get cancelled leaving the first and the last term alone.

$$\frac{-1}{4} + \frac{(-1)^n}{4} \left(\frac{1}{(2n+1)} \right)$$

Which is the required solution.

Verification

$$\sum_{k=1}^{n} \frac{(-1)^k\, k}{(4k^2 - 1)} = \frac{-1}{4} + \frac{(-1)^n}{4}\left(\frac{1}{(2n+1)}\right)$$

n	1	2	3	4
$\sum_{k=1}^{n}(((-1)^k k)/(4k^2-1))$	-1/3	-1/5	-2/7	-2/9
$(-1/4)+(((1)^n/4)(1/(2n+1))$	-1/3	-1/5	-2/7	-2/9