CSE547
chapter 2
Problems 20,21
Try to evaluate $\sum_{0 \leq k \leq n} kH_k$ by the perturbation method, but deduce the value of $\sum_{0 \leq k \leq n} H_k$ instead.
Perturbation Method

\[ S_n = \sum_{0 \leq k \leq n} a_k \]

\[ S_n + a_{n+1} = \sum_{0 \leq k \leq n+1} a_k = a_0 + \sum_{1 \leq k \leq n+1} a_k \]

\[ = a_0 + \sum_{1 \leq k+1 \leq n+1} a_{k+1} \]

\[ = a_0 + \sum_{0 \leq k \leq n} a_{k+1} \]
Harmonic Number

\[ H_n = 1 + \frac{1}{2} + \ldots + \frac{1}{n} = \sum_{1 \leq k \leq n} \frac{1}{k} \]

Ex 20) \[ S_n = \sum_{0 \leq k \leq n} kH_k \]

\[ S_n + (n+1)H_{n+1} = \sum_{0 \leq k \leq n+1} kH_k = 0H_0 + \sum_{1 \leq k \leq n+1} kH_k \]

\[ = 0 + \sum_{1 \leq k+1 \leq n+1} (k+1)H_{k+1} \]

\[ = \sum_{0 \leq k \leq n} (k+1)H_{k+1} \]
\[ H_n = \sum_{1 \leq k \leq n} \frac{1}{k} \]

\[ H_{n+1} = \sum_{1 \leq k \leq n} \frac{1}{k} + \frac{1}{n+1} \]

\[ S_n + (n+1)H_{n+1} = \sum_{0 \leq k \leq n} (k+1)H_{k+1} \]

\[ = \sum_{0 \leq k \leq n} (k+1)(\frac{1}{k+1} + H_k) \]

\[ = \sum_{0 \leq k \leq n} 1 + \sum_{0 \leq k \leq n} kH_k + \sum_{0 \leq k \leq n} H_k \]
\[ S_n + (n + 1)H_{n+1} = \sum_{0 \leq k \leq n} 1 + \sum_{0 \leq k \leq n} kH_k + \sum_{0 \leq k \leq n} H_k \]

\[ (n + 1)H_{n+1} = \sum_{0 \leq k \leq n} 1 + \sum_{0 \leq k \leq n} H_k \]

\[ \sum_{0 \leq k \leq n} H_k = (n + 1)H_{n+1} - \sum_{0 \leq k \leq n} 1 \]

\[ \sum_{0 \leq k \leq n} H_k = (n + 1)H_{n+1} - (n + 1) \quad \text{//End of ex.20} \]
Perturbation Method

Excercises 21

Evaluate by perturbation method

\[ S_n = \sum_{k=0}^{n} (-1)^{n-k}, \quad T_n = \sum_{k=0}^{n} (-1)^{n-k} k, \]

and \[ U_n = \sum_{k=0}^{n} (-1)^{n-k} k^2 \]

Assume \( n \geq 0 \)
\[ S_n = \sum_{0 \leq k \leq n} (-1)^{n-k} \]

Which implies,

\[ S_{n+1} = \sum_{0 \leq k \leq n+1} (-1)^{n+1-k} \]

Split off the first term

\[ S_{n+1} = \sum_{0 \leq k \leq n+1} (-1)^{n+1-k} \]

\[ = a_0 + \sum_{1 \leq k \leq n+1} (-1)^{n+1-k} \]

\[ = (-1)^{n+1-0} + \sum_{1 \leq k+1 \leq n+1} (-1)^{(n+1)-(k+1)} \]

\[ = (-1)^{n+1} + \sum_{0 \leq k \leq n} (-1)^{n-k} \]

\[ = (-1)^{n+1} + S_n \ldots \ldots \ldots \ldots (a1) \]
\[ S_n = \sum_{0 \leq k \leq n} (-1)^{n-k} \]

Split off the final term

\[ S_{n+1} = \sum_{0 \leq k \leq n+1} (-1)^{(n+1)-k} \]

\[ = \sum_{0 \leq k \leq n} (-1)^{n+1-k} + (-1)^{(n+1)-(n+1)} \]

\[ = \sum_{0 \leq k \leq n} (-1)^{n+1-k} + 1 \]

\[ = -\sum_{0 \leq k \leq n} (-1)^{n-k} + 1 \]

\[ = 1 - S_n \] \((a2)\)
Equating (a1) and (a2) we get….

\[ 1 - S_n = (-1)^{n+1} + S_n \]

\[ \Rightarrow 2S_n = 1 - (-1)^{n+1} \]

\[ \Rightarrow S_n = \frac{1 + (-1)^n}{2} \]
\[ T_n = \sum_{k=0}^{n} (-1)^{n-k} k \]

Which implies

\[ T_{n+1} = \sum_{k=0}^{n+1} (-1)^{n+1-k} k \]

Split off the first term

\[ T_{n+1} = \sum_{k=0}^{n+1} (-1)^{n+1-k} k \]

\[ = 0\cdot(-1)^{(n+1)} + \sum_{k=1}^{n+1} (-1)^{n+1-k} k \]

\[ = \sum_{1 \leq k+1 \leq n+1} (-1)^{n+1-(k+1)} (k + 1) \]

\[ = \sum_{0 \leq k \leq n} (-1)^{n-k} (k + 1) \]

\[ = \sum_{0 \leq k \leq n} (-1)^{n-k} k + \sum_{0 \leq k \leq n} (-1)^{n-k} \]

\[ = T_n + S_n \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots (b1) \]
Split off the final term

\[ T_{n+1} = \sum_{k=0}^{n+1} (-1)^{n+1-k} k \]

\[ = \sum_{k=0}^{n} (-1)^{n+1-k} k + (-1)^{n+1-(n+1)} (n + 1) \]

\[ = -\sum_{k=0}^{n} (-1)^{n-k} k + (n + 1) \]

\[ = (n + 1) - T_n \]

\[ \text{...............}(b2) \]
Equating (b1) and (b2)……

\[
(n + 1) - T_n = T_n + S_n
\]

\[
\Rightarrow 2T_n = (n + 1) - (1 + (-1)^n)
\]

\[
\Rightarrow T_n = \frac{1}{2}(n - (-1)^n)
\]
\[ U_n = \sum_{k=0}^{n} (-1)^{n-k} k^2 \]

Which implies

\[ U_{n+1} = \sum_{k=0}^{n+1} (-1)^{n+1-k} k^2 \]

Split off the first term

\[ U_{n+1} = 0.(-1)^{(n+1)-0} + \sum_{k=1}^{n+1} (-1)^{n+1-k} k^2 \]

\[ = \sum_{k=1}^{n+1} (-1)^{n+1-(k+1)} (k+1)^2 \]

\[ = \sum_{k=0}^{n} (-1)^{-k} (k^2 + 2k + 1) \]

\[ = \sum_{k=0}^{n} (-1)^{-k} k^2 + \sum_{k=0}^{n} (-1)^{-k} 2k + \sum_{k=0}^{n} (-1)^{-k} \]

\[ = U_n + 2T_n + S_n \text{..................................(c1)} \]
Split off the final term

\[ U_{n+1} = \sum_{k=0}^{n} (-1)^{n+1-k} k^2 + (-1)^{n+1-(n+1)} (n + 1)^2 \]

\[ = - \sum_{k=0}^{n} (-1)^{n-k} k^2 + (n + 1)^2 \]

\[ = -U_n + (n + 1)^2 \] ...............................(c2)
Equating (c1) and (c2)……

\[(n+1)^2 - U_n = U_n + 2T_n + S_n\]

\[\Rightarrow 2U_n = (n+1)^2 - 2T_n - S_n\]

\[\Rightarrow 2U_n = (n+1)^2 - (n - (-1)^n) - (1 + (-1)^n)\]

\[\Rightarrow U_n = \frac{n^2 + n}{2}\]