# Ch2 Problem 16

Prove that

$$\frac{x^{\underline{m}}}{(x-n)^{\underline{m}}} = \frac{x^{\underline{n}}}{(x-m)^{\underline{n}}}$$

unless one of the denominators is zero.

# Definition of $x^{\underline{n}}$

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When $n > 0$

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When $n < 0$

$$x^{\underline{n}} = \frac{1}{(x+1)(x+2)\cdots(x-n)}$$

When $n = 0$

$$x^{\underline{n}} = 1$$

# Proof 1 – Case 1: $n = 0$ or $m = 0$

$$\frac{x^{\underline{m}}}{(x-n)^{\underline{m}}} = \frac{x^{\underline{n}}}{(x-m)^{\underline{n}}}$$

Trivial.

# Proof 1 – Case 2: $n > 0$ and $m > 0$

Use definition $x^{\underline{n}} = \frac{x!}{(x-n)!}$

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Use definition $x^{\underline{n}} = \frac{x!}{(x-n)!}$

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So

$$\frac{x^{\underline{m}}}{(x-n)^{\underline{m}}} = \frac{x^{\underline{n}}}{(x-m)^{\underline{n}}}$$

# Proof 1 – Case 3: $n < 0$ and $m < 0$

Use definition $x^{\underline{n}} = \frac{1}{(x+1)(x+2)\cdots(x-n)}$

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Without losing generality, assume $n \leq m < 0$ so that $0 < -m \leq -n$

# Proof 1 – Case 3: $n < 0$ and $m < 0$

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Without losing generality, assume $n \leq m < 0$ so that $0 < -m \leq -n$

$$\begin{aligned}\frac{x^{\underline{n}}}{(x-m)^{\underline{n}}} &= \frac{(x-m+1)\cdots(x-m-n)}{(x+1)\cdots(x-n)} \\ &= \frac{(x-m+1)\cdots(x-n)(x-n+1)\cdots(x-m-n)}{(x+1)\cdots(x-m)(x-m+1)\cdots(x-n)} \\ &= \frac{x^{\underline{m}}}{(x-n)^{\underline{m}}}\end{aligned}$$

# Proof 1 – Case 4: $n < 0$ and $m > 0$

If $-m - n \geq 1$

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If $-m - n \geq 1$

$$\frac{x^{\underline{m}}}{(x-n)^{\underline{m}}} = \frac{x(x-1)\cdots(x-m+1)}{(x-n)(x-n-1)\cdots(x-n-m+1)}$$

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If $-m - n \geq 1$

$$\frac{x^{\underline{m}}}{(x-n)^{\underline{m}}} = \frac{x(x-1)\cdots(x-m+1)}{(x-n)(x-n-1)\cdots(x-n-m+1)}$$

$$\begin{aligned}\frac{x^{\underline{n}}}{(x-m)^{\underline{n}}} &= \frac{\frac{1}{(x+1)\cdots(x-n)}}{\frac{1}{(x-m+1)\cdots(x-m-n)}} \\ &= \frac{(x-m+1)\cdots(x-m-n)}{(x+1)\cdots(x-n)} \\ &= \frac{(x-m+1)\cdots x(x+1)\cdots(x-m-n)}{(x+1)\cdots(x-m-n)(x-m-n+1)\cdots(x-n)} \\ &= \frac{x^{\underline{m}}}{(x-n)^{\underline{m}}}\end{aligned}$$

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# Proof 1 – Case 4: $n < 0$ and $m > 0$

If $-m - n \leq -1$

$$\frac{x^{\underline{n}}}{(x-m)^{\underline{n}}} = \frac{(x-m+1)\cdots(x-m-n)}{(x+1)\cdots(x-n)}$$

$$\begin{aligned}
\frac{x^{\underline{m}}}{(x-n)^{\underline{m}}} &= \frac{x(x-1)\cdots(x-m+1)}{(x-n)(x-n-1)\cdots(x-n-m+1)} \\
&= \frac{x\cdots(x-n-m+1)(x-n-m)\cdots(x-m+1)}{(x-n)\cdots(x+1)x\cdots(x-n-m+1)} \\
&= \frac{(x-m+1)\cdots(x-m-n)}{(x+1)\cdots(x-n)} \\
&= \frac{x^{\underline{n}}}{(x-m)^{\underline{n}}}
\end{aligned}$$

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If $-1 < -m - n < 1 \Rightarrow -m - n = 0$

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$$\frac{x^{\underline{n}}}{(x-m)^{\underline{n}}} = \frac{(x-m+1)\cdots(x-m-n)}{(x+1)\cdots(x-n)}$$

$$= \frac{(x-m+1)\cdots x}{(x-n-m+1)\cdots(x-n)}$$

$$= \frac{x^{\underline{m}}}{(x-n)^{\underline{m}}}$$

# Proof 1 – Case 5: $n > 0$ and $m < 0$

Symmetrical to Case 4.

# Proof 2

**Theorem.**

$$x^{\underline{m+n}} = x^{\underline{m}}(x-m)^{\underline{n}}$$

*for all the integer $m$ and $n$.*

By the theorem above, we directly have:

$$x^{\underline{m}}(x-m)^{\underline{n}} = x^{\underline{m+n}} = x^{\underline{n}}(x-n)^{\underline{m}}$$

So

$$\frac{x^{\underline{m}}}{(x-n)^{\underline{m}}} = \frac{x^{\underline{n}}}{(x-m)^{\underline{n}}}$$

# Ch2 Problem 17

Prove that

$$x^{\overline{m}} = (-1)^m(-x)^{\underline{m}} = (x+m-1)^{\underline{m}} = \frac{1}{(x-1)^{\underline{-m}}}$$

$$x^{\underline{m}} = (-1)^m(-x)^{\overline{m}} = (x-m+1)^{\overline{m}} = \frac{1}{(x+1)^{\overline{-m}}}$$

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When $n = 0$

$$x^{\overline{n}} = 1$$

# Proof – Case 1: $m = 0$

$$x^{\overline{m}} = (-1)^m(-x)^{\underline{m}} = (x+m-1)^{\underline{m}} = \frac{1}{(x-1)^{\underline{-m}}}$$

$$x^{\underline{m}} = (-1)^m(-x)^{\overline{m}} = (x-m+1)^{\overline{m}} = \frac{1}{(x+1)^{\overline{-m}}}$$

Trivial.

# Proof – Case 2: $m > 0$

Definitions recap:

$$x^{\underline{n}} = \begin{cases} x(x-1)(x-2)\cdots(x-n+1) & n > 0 \\ 1/(x+1)(x+2)\cdots(x-n) & n < 0 \\ 1 & n = 0 \end{cases}$$

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# Proof – Case 2: $m > 0$

Totaly by definitions shown in the last slide.

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$$x^{\overline{m}} = x(x+1)(x+2)\cdots(x+m-1)$$

$$(-1)^m(-x)^{\underline{m}} = (-1)^m(-x)(-x-1)\cdots(-x-m+1)$$

$$= x(x+1)\cdots(x+m-1)$$

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$$= x(x+1)\cdots(x+m-1)$$

$$(x+m-1)^{\underline{m}} = (x+m-1)\cdots(x+1)x$$

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$$x^{\overline{m}} = x(x+1)(x+2)\cdots(x+m-1)$$

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$$(x+m-1)^{\underline{m}} = (x+m-1)\cdots(x+1)x$$

$$\begin{aligned}\frac{1}{(x-1)^{\underline{-m}}} &= (x-1+1)(x-1+2)\cdots(x-1+m)\\ &= x(x+1)\cdots(x+m-1)\end{aligned}$$

# Proof – Case 3: $m < 0$

Definitions recap:

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$$x^{\overline{m}} = 1/(x-1)(x-2)\cdots(x+m)$$

$$(-1)^m(-x)^{\underline{m}} = (-1)^m/(-x+1)(-x+2)\cdots(-x-m)$$
$$= 1/(x-1)(x-2)\cdots(x+m)$$

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$$\begin{aligned}(x+m-1)^{\underline{m}} &= 1/(x+m-1+1)(x+m-1+2)\cdots(x+m-1-m\\ &= 1/(x-1)(x-2)\cdots(x+m)\end{aligned}$$

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$$\begin{aligned}\frac{1}{(x-1)^{\underline{-m}}} &= 1/(x-1)(x-1-1)\cdots(x-1+m-1)\\ &= 1/(x-1)(x-2)\cdots(x+m)\end{aligned}$$

# Solution – The other equation

Similarly, we can easily prove the other equation:

$$x^{\underline{m}} = (-1)^m(-x)^{\overline{m}} = (x-m+1)^{\overline{m}} = \frac{1}{(x+1)^{\overline{-m}}}$$

# Thank You!

*Any Questions?*