Ch2 Problem 16

Prove that

\[
\frac{x^m}{(x - n)^m} = \frac{x^n}{(x - m)^n}
\]

unless one of the denominators is zero.
Definition of $x^n$
Definition of $x^n$

When $n > 0$
Definition of $x^n$

When $n > 0$

$$x^n = x(x - 1)(x - 2) \cdots (x - n + 1) = \frac{x!}{(x - n)!}$$
Definition of $x^n$

When $n > 0$

$$x^n = x(x - 1)(x - 2) \cdots (x - n + 1) = \frac{x!}{(x - n)!}$$

When $n < 0$
Definition of $x^n$

When $n > 0$

$$x^n = x(x - 1)(x - 2) \cdots (x - n + 1) = \frac{x!}{(x - n)!}$$

When $n < 0$

$$x^n = \frac{1}{(x + 1)(x + 2) \cdots (x - n)}$$
Definition of $x^n$

When $n > 0$

$$x^n = x(x - 1)(x - 2) \cdots (x - n + 1) = \frac{x!}{(x - n)!}$$

When $n < 0$

$$x^n = \frac{1}{(x + 1)(x + 2) \cdots (x - n)}$$

When $n = 0$
Definition of $x^n$

When $n > 0$

$$x^n = x(x - 1)(x - 2) \cdots (x - n + 1) = \frac{x!}{(x - n)!}$$

When $n < 0$

$$x^n = \frac{1}{(x + 1)(x + 2) \cdots (x - n)}$$

When $n = 0$

$$x^n = 1$$
Proof 1 – Case 1: $n = 0$ or $m = 0$

$$\frac{x^m}{(x - n)^m} = \frac{x^n}{(x - m)^n}$$

Trivial.
Proof 1 – Case 2: \( n > 0 \) and \( m > 0 \)

Use definition \( x^n = \frac{x!}{(x-n)!} \)
Proof 1 – Case 2: $n > 0$ and $m > 0$

Use definition $x^n = \frac{x!}{(x-n)!}$

$$\frac{x^n}{(x-n)^m} = \frac{x!}{(x-m)!} = \frac{x!}{(x-n)! \cdot (x-n-m)!} = \frac{x!(x-n-m)!}{(x-n)!(x-m)!}$$
Proof 1 – Case 2: $n > 0$ and $m > 0$

Use definition $x^n = \frac{x!}{(x-n)!}$

\[
\frac{x^m}{(x-n)^m} = \frac{x!}{(x-m)!} = \frac{x!(x-n-m)!}{(x-n)!}{(x-m)!}
\]

\[
\frac{x^n}{(x-m)^n} = \frac{x!}{(x-n)!} = \frac{x!(x-n-m)!}{(x-n)!}{(x-m)!}
\]
Proof 1 – Case 2: $n > 0$ and $m > 0$

Use definition $x^n = \frac{x!}{(x-n)!}$

\[
\frac{x^m}{(x - n)^m} = \frac{x!}{(x-m)!} = \frac{x!(x - n - m)!}{(x - n)!(x - m)!}
\]

\[
\frac{x^n}{(x - m)^n} = \frac{x!}{(x-m)!} = \frac{x!(x - n - m)!}{(x - n)!(x - m)!}
\]

So

\[
\frac{x^m}{(x - n)^m} = \frac{x^n}{(x - m)^n}
\]
Proof 1 – Case 3: \( n < 0 \) and \( m < 0 \)

Use definition \( x^n = \frac{1}{(x+1)(x+2)\cdots(x-n)} \)
Proof 1 – Case 3: $n < 0$ and $m < 0$

Use definition $x^n = \frac{1}{(x+1)(x+2)\cdots(x-n)}$

$$\frac{x^m}{(x-n)^m} = \frac{1}{(x+1)\cdots(x-m)} \cdot \frac{1}{(x-n+1)\cdots(x-n-m)} = \frac{(x-n+1)\cdots(x-n-m)}{(x+1)\cdots(x-m)}$$
Proof 1 – Case 3: $n < 0$ and $m < 0$

Use definition $x^n = \frac{1}{(x+1)(x+2)\cdots(x-n)}$

\[
\frac{x^m}{(x-n)^m} = \frac{1}{(x+1)\cdots(x-m)} = \frac{(x-n+1)\cdots(x-n-m)}{(x+1)\cdots(x-m)}
\]

\[
\frac{x^n}{(x-m)^n} = \frac{1}{(x+1)\cdots(x-n)} = \frac{(x-m+1)\cdots(x-m-n)}{(x+1)\cdots(x-n)}
\]
Proof 1 – Case 3: $n < 0$ and $m < 0$

Use definition $x^n = \frac{1}{(x+1)(x+2)\cdots(x-n)}$

$$\frac{x^m}{(x-n)^m} = \frac{1}{(x+1)\cdots(x-m)} = \frac{(x-n+1)\cdots(x-n-m)}{(x+1)\cdots(x-m)}$$

$$\frac{x^n}{(x-m)^n} = \frac{1}{(x+1)\cdots(x-n)} = \frac{(x-m+1)\cdots(x-m-n)}{(x+1)\cdots(x-n)}$$

Without losing generality, assume $n \leq m < 0$ so that $0 < -m \leq -n$
Proof 1 – Case 3: \( n < 0 \) and \( m < 0 \)

Use definition \( x^n = \frac{1}{(x+1)(x+2)\cdots(x-n)} \)

\[
\frac{x^m}{(x-n)^m} = \frac{1}{(x+1)\cdots(x-m)} = \frac{(x-n+1)\cdots(x-n-m)}{(x+1)\cdots(x-m)}
\]

\[
\frac{x^n}{(x-m)^n} = \frac{1}{(x+1)\cdots(x-n)} = \frac{(x-m+1)\cdots(x-m-n)}{(x+1)\cdots(x-n)}
\]

Without losing generality, assume \( n \leq m < 0 \) so that \( 0 < -m \leq -n \)

\[
\frac{x^n}{(x-m)^n} = \frac{(x - m + 1) \cdots (x - m - n)}{(x + 1) \cdots (x - n)}
\]

\[
= \frac{(x - m + 1) \cdots (x - n)(x - n + 1) \cdots (x - m - n)}{(x + 1) \cdots (x - m)(x - m + 1) \cdots (x - n)}
\]

\[
= \frac{x^m}{(x - n)^m}
\]
Proof 1 – Case 4: \( n < 0 \) and \( m > 0 \)

If \(-m - n \geq 1\)
Proof 1 – Case 4: \( n < 0 \) and \( m > 0 \)

If \(-m - n \geq 1\)

\[
\frac{x^m}{(x - n)^m} = \frac{x(x - 1) \cdots (x - m + 1)}{(x - n)(x - n - 1) \cdots (x - n - m + 1)}
\]
Proof 1 – Case 4: $n < 0$ and $m > 0$

If $-m - n \geq 1$

\[
\frac{x^m}{(x - n)^m} = \frac{x(x - 1) \cdots (x - m + 1)}{(x - n)(x - n - 1) \cdots (x - n - m + 1)}
\]

\[
\frac{x^n}{(x - m)^n} = \frac{1}{(x+1) \cdots (x-n)} \frac{1}{(x-m+1) \cdots (x-m-n)}
\]

\[
= \frac{(x - m + 1) \cdots (x - m - n)}{(x + 1) \cdots (x - n)}
\]

\[
= \frac{(x - m + 1) \cdots x(x + 1) \cdots (x - m - n)}{(x + 1) \cdots (x - m - n)(x - m - n + 1) \cdots (x - n)}
\]

\[
= \frac{x^m}{(x - n)^m}
\]
Proof 1 – Case 4: $n < 0$ and $m > 0$

If $-m - n \leq -1$
Proof 1 – Case 4: \( n < 0 \) and \( m > 0 \)

If \(-m - n \leq -1\)

\[
\frac{x^n}{(x - m)^n} = \frac{(x - m + 1) \cdots (x - m - n)}{(x + 1) \cdots (x - n)}
\]
Proof 1 – Case 4: $n < 0$ and $m > 0$

If $-m - n \leq -1$

\[
\frac{x^n}{(x - m)^n} = \frac{(x - m + 1) \cdots (x - m - n)}{(x + 1) \cdots (x - n)}
\]

\[
\frac{x^m}{(x - n)^m} = \frac{x(x - 1) \cdots (x - m + 1)}{(x - n)(x - n - 1) \cdots (x - n - m + 1)}
\]

\[
= \frac{x \cdots (x - n - m + 1)(x - n - m) \cdots (x - m + 1)}{(x - n) \cdots (x + 1)x \cdots (x - n - m + 1)}
\]

\[
= \frac{(x - m + 1) \cdots (x - m - n)}{(x + 1) \cdots (x - n)}
\]

\[
= \frac{x^n}{(x - m)^n}
\]
Proof 1 – Case 4: \( n < 0 \) and \( m > 0 \)

If \(-1 < -m - n < 1 \) \( \Rightarrow -m - n = 0 \)
Proof 1 – Case 4: \( n < 0 \) and \( m > 0 \)

If \(-1 < -m - n < 1 \) \( \Rightarrow -m - n = 0 \)

\[
\frac{x^n}{(x - m)^n} = \frac{(x - m + 1) \cdots (x - m - n)}{(x + 1) \cdots (x - n)}
\]

\[
= \frac{(x - m + 1) \cdots x}{(x - n - m + 1) \cdots (x - n)}
\]

\[
= \frac{x^m}{(x - n)^m}
\]
Proof 1 – Case 5: $n > 0$ and $m < 0$

Symmetrical to Case 4.
Proof 2

Theorem.

\[ x^{m+n} = x^m(x - m)^n \]

for all the integer \( m \) and \( n \).

By the theorem above, we directly have:

\[ x^m(x - m)^n = x^{m+n} = x^n(x - n)^m \]

So

\[ \frac{x^m}{(x - n)^m} = \frac{x^n}{(x - m)^n} \]
Prove that

\[ x^m = (-1)^m (-x)^m = (x + m - 1)^m = \frac{1}{(x - 1)^{-m}} \]

\[ x^m = (-1)^m (-x)^m = (x - m + 1)^m = \frac{1}{(x + 1)^{-m}} \]
Definition of $x^n$
Definition of $x^n$

When $n > 0$
Definition of $x^n$

When $n > 0$

$$x^n = x(x + 1)(x + 2) \cdots (x + n - 1)$$
Definition of $x^n$

When $n > 0$

$$x^n = x(x + 1)(x + 2) \cdots (x + n - 1)$$

When $n < 0$
Definition of $x^n$

When $n > 0$

$$x^n = x(x + 1)(x + 2) \cdots (x + n - 1)$$

When $n < 0$

$$x^n = \frac{1}{(x - 1)(x - 2) \cdots (x + n)}$$
Definition of $x^n$

When $n > 0$

$$x^n = x(x + 1)(x + 2) \cdots (x + n - 1)$$

When $n < 0$

$$x^n = \frac{1}{(x - 1)(x - 2) \cdots (x + n)}$$

When $n = 0$
Definition of $x^n$

When $n > 0$

$$x^n = x(x + 1)(x + 2) \cdots (x + n - 1)$$

When $n < 0$

$$x^n = \frac{1}{(x - 1)(x - 2) \cdots (x + n)}$$

When $n = 0$

$$x^n = 1$$
Proof – Case 1: \( m = 0 \)

\[
\begin{align*}
\bar{x}^m &= (-1)^m (-x)^m = (x + m - 1)^m = \frac{1}{(x - 1)^{-m}} \\
\overline{x^m} &= (-1)^m (-x)^\overline{m} = (x - m + 1)^\overline{m} = \frac{1}{(x + 1)^{-m}}
\end{align*}
\]

Trivial.
Proof – Case 2: $m > 0$

Definitions recap:

$$x^n = \begin{cases} 
  x(x-1)(x-2) \cdots (x-n+1) & n > 0 \\
  1/(x+1)(x+2) \cdots (x-n) & n < 0 \\
  1 & n = 0 
\end{cases}$$

$$x^n = \begin{cases} 
  x(x+1)(x+2) \cdots (x+n-1) & n > 0 \\
  1/(x-1)(x-2) \cdots (x+n) & n < 0 \\
  1 & n = 0 
\end{cases}$$
Proof – Case 2: $m > 0$

Totally by definitions shown in the last slide.
Proof – Case 2: \( m > 0 \)

Totally by definitions shown in the last slide.

\[
x^m = x(x + 1)(x + 2) \cdots (x + m - 1)
\]
Proof – Case 2: $m > 0$

Totally by definitions shown in the last slide.

$$x^m = x(x + 1)(x + 2) \cdots (x + m - 1)$$

$$(-1)^m(-x)^m = (-1)^m(-x)(-x - 1) \cdots (-x - m + 1)$$

$$= x(x + 1) \cdots (x + m - 1)$$
Proof – Case 2: \( m > 0 \)

Totaly by definitions shown in the last slide.

\[
x^m = x(x + 1)(x + 2) \cdots (x + m - 1)
\]

\[
(-1)^m(-x)^m = (-1)^m(-x)(-x - 1) \cdots (-x - m + 1)
\]

\[
= x(x + 1) \cdots (x + m - 1)
\]

\[
(x + m - 1)^m = (x + m - 1) \cdots (x + 1)x
\]
Proof – Case 2: \( m > 0 \)

Totaly by definitions shown in the last slide.

\[
x^m = x(x + 1)(x + 2) \cdots (x + m - 1)
\]

\[
(-1)^m(-x)^m = (-1)^m(-x)(-x - 1) \cdots (-x - m + 1)
= x(x + 1) \cdots (x + m - 1)
\]

\[
(x + m - 1)^m = (x + m - 1) \cdots (x + 1)x
\]

\[
\frac{1}{(x - 1)^{-m}} = (x - 1 + 1)(x - 1 + 2) \cdots (x - 1 + m)
= x(x + 1) \cdots (x + m - 1)
\]
Proof – Case 3: $m < 0$

Definitions recap:

$$x^n = \begin{cases} 
  x(x-1)(x-2)\cdots(x-n+1) & n > 0 \\
  \frac{1}{(x+1)(x+2)\cdots(x-n)} & n < 0 \\
  1 & n = 0 
\end{cases}$$

$$\overline{x}^n = \begin{cases} 
  x(x+1)(x+2)\cdots(x+n-1) & n > 0 \\
  \frac{1}{(x-1)(x-2)\cdots(x+n)} & n < 0 \\
  1 & n = 0 
\end{cases}$$
Proof – Case 3: $m < 0$

Totaly by definitions shown in the last slide.
Proof – Case 3: \( m < 0 \)

Totally by definitions shown in the last slide.

\[ x^m = \frac{1}{(x - 1)(x - 2) \cdots (x + m)} \]
Proof – Case 3: \( m < 0 \)

Totally by definitions shown in the last slide.

\[
x^{-m} = 1/(x - 1)(x - 2) \cdots (x + m)
\]

\[
(-1)^m (-x)^m = (-1)^m/(-x + 1)(-x + 2) \cdots (-x - m)
\]

\[
= 1/(x - 1)(x - 2) \cdots (x + m)
\]
Proof – Case 3: $m < 0$

Totaly by definitions shown in the last slide.

$$x^m = 1/(x - 1)(x - 2) \cdots (x + m)$$

$$(-1)^m (-x)^m = (-1)^m/(-x + 1)(-x + 2) \cdots (-x - m)$$

$$= 1/(x - 1)(x - 2) \cdots (x + m)$$

$$(x + m - 1)^m = 1/(x + m - 1 + 1)(x + m - 1 + 2) \cdots (x + m - 1 - m)$$

$$= 1/(x - 1)(x - 2) \cdots (x + m)$$
Proof – Case 3: \( m < 0 \)

Totaly by definitions shown in the last slide.

\[
x^m = \frac{1}{(x - 1)(x - 2) \cdots (x + m)}
\]

\[
(-1)^m (-x)^m = \frac{(-1)^m}{(-x + 1)(-x + 2) \cdots (-x - m)}
= \frac{1}{(x - 1)(x - 2) \cdots (x + m)}
\]

\[
(x + m - 1)^m = \frac{1}{(x + m - 1 + 1)(x + m - 1 + 2) \cdots (x + m - 1 - m)}
= \frac{1}{(x - 1)(x - 2) \cdots (x + m)}
\]

\[
\frac{1}{(x - 1)^{-m}} = \frac{1}{(x - 1)(x - 1 - 1) \cdots (x - 1 + m - 1)}
= \frac{1}{(x - 1)(x - 2) \cdots (x + m)}
\]
Solution – The other equation

Similarly, we can easily prove the other equation:

\[ x^m = (-1)^m (-x)^m = (x - m + 1)^m = \frac{1}{(x + 1)^{-m}} \]
Thank You!

Any Questions?