<u>CSE547</u>

Chapter 2
Problems 13 and 14

Problem 13

Use the Repertoire method to find a closed form for:

$$\sum_{k=0}^{n} (-1)^2 k^2$$

Problem 13: Solution

$$S_n = \sum_{k=0}^{n} (-1)^k k^2 \qquad \text{(for all } n \in \mathbb{N}\text{)}$$

$$S_0 = 0$$

$$S_n = S_{n-1} + (-1)^n n^2$$

Problem 13: Solution

Generalizing this expression in a recursive form

$$\mathbf{R_0 = \alpha}$$

$$\mathbf{R_n = R_{n-1} + (-1)^n (\beta + n\gamma + n^2\delta)} \quad \ldots\ldots\ldots\ldots\ldots\ldots(1)$$

A Special case of with : $\alpha = 0$, $\beta = 0$, $\gamma = 0$, $\delta = 1$

Problem 13: Solution

To get a solution in the closed form, we express R_n in a generalized form. We write R_n as:

$$R_n = A(n)\alpha + B(n)\beta + C(n)\gamma + D(n)\delta$$

We need to solve this equation to find the values of A(n), B(n), C(n) and D(n)

Problem 13: Solution
Steps to solve:

- Pick simple functions (R_n) with easy values for α, β, γ, δ and the solve the equation to find value of A(n), B(n), C(n) and D(n).
- Substitute the values of A(n), B(n), C(n) and D(n) to get the general equation for the recurrence.

Case 1: Taking $R_n = 1$ (for all n Є N)

$R_0 = \alpha$

$R_0 = 1$ .... So, $R_0 = \alpha = 1$

$R_n = R_{n-1} + (-1)^n (\beta + \gamma n + \delta n^2)$

$1 = 1 + (-1)^n (\beta + \gamma n + \delta n^2)$

$0 = (-1)^n (\beta + \gamma n + \delta n^2)$

$\beta + n\gamma + n^2\delta = 0$ (dividing both sides by $(-1)^n$)

$(\beta - 0) + n(\gamma - 0) + n^2 (\delta - 0) = 0$

So, $\beta = 0$, $\gamma = 0$ and $\delta = 0$

Putting these values in $R_n = A(n)\alpha + B(n)\beta + C(n)\gamma + D(n)\delta$, we have

$R_n = A(n)\, 1 + 0$

Since $R_n = 1$, **$A(n) = 1$** (for all n Є N)...**.......(2)**

Case 2: Taking $R_n = n$ (for all $n \in N$)

$R_0 = \alpha$

$R_0 = 0$ ………… So $\alpha = 0$

$R_n = R_{n-1} + (-1)^n (\beta + \gamma n + \delta n^2)$

$n = n - 1 + (-1)^n (\beta + \gamma n + \delta n^2)$

$1 = (-1)^n (\beta + \gamma n + \delta n^2)$

$(-1)^{-n} = \beta + \gamma n + \delta n^2$

We cannot solve this further.

The recurrence is not found to be true for this case.

Therefore, we try with another value of R_n

Case 3: Taking $R_n = (-1)^n$ (for all n Є N)

$R_0 = \alpha$

$R_0 = (-1)^0$ ………… So $\alpha = 1$

$R_n = R_{n-1} + (-1)^n (\beta + \gamma n + \delta n^2)$

$(-1)^n = (-1)^{(n-1)} + (-1)^n (\beta + \gamma n + \delta n^2)$

dividing both sides by $(-1)^{n-1}$

$(-1) = 1 + (-1)(\beta + \gamma n + \delta n^2)$

$(-2) = (-1)(\beta + \gamma n + \delta n^2)$

Rearranging the terms,

$(\beta - 2) + (\gamma - 0)n + (\delta - 0)n^2 = 0$

So we have $\beta = 2$, $\gamma = 0$ and $\delta = 0$

Putting these values in $R_n = A(n)\alpha + B(n)\beta + C(n)\gamma + D(n)\,\delta$, we have

$R_n = A(n)\,1 + B(n)\,2 + C(n)\,0 + D(n)\,0 = (-1)^n$

$2\,B(n) + 1 = (-1)^n$ ...Since $A(n) = 1$ (for all $n \in N$)

$B(n) = ((-1)^n - 1)/2$ (for all $n \in N$) **..........(3)**

Case 4: Taking $R_n = (-1)^n n$ (for all n Є N)

$R_0 = \alpha$

$R_0 = 0$ ………………..So $\alpha = 0$

$$R_n = R_{n-1} + (-1)^n (\beta + \gamma n + \delta n^2)$$

$$(-1)^n n = (-1)^{(n-1)} (n - 1) + (-1)^n (\beta + \gamma n + \delta n^2)$$

$$(-1) n = (n - 1) + (-1) (\beta + \gamma n + \delta n^2)$$

$$0 = (2n - 1) - \beta - \gamma n - \delta n^2$$

$$0 = (-1 - \beta) - n (2 - \gamma) - \delta n^2$$

So $\beta = -1, \gamma = 2, \delta = 0$

$$R_n = A(n) + B(n)\beta + C(n)\gamma + D(n)\delta$$

Substituting $\alpha = 0, \beta = -1, \gamma = 2, \delta = 0$

$$(-1)^n n = 0 + B(n)(-1) + C(n)(2) + 0$$

$$\mathbf{(-1)^n n = -B(n) + 2C(n) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(4)}$$

Substituting $B(n) = ((-1)^n - 1)/2$ (for all $n \in N$)

$$(-1)^n n = -(((-1)^n - 1)/2) + 2C(n)$$

$$2C(n) = (((-1)^n(2n+1)) - 1)/2$$

$$\mathbf{C(n) = (((-1)^n(2n+1)) - 1)/4}$$

(for all $n \in N$) ...**(5)**

Case 5 : Taking $R_n = (-1)^n n^2$

(for all $n \in N$)

$R_0 = \alpha$

$R_0 = 0$ ....So $\alpha = 0$

$$R_n = R_{n-1} + (-1)^n (\beta + \gamma n + \delta n^2)$$

$$(-1)^n n^2 = (-1)^{(n-1)} (n-1)^2 + (-1)^n (\beta + \gamma n + \delta n^2)$$

$$(-1) n^2 = (n-1)^2 + (-1) (\beta + \gamma n + \delta n^2)$$

$$0 = 2n^2 - 2n + 1 - \beta - \gamma n - \delta n^2$$

$$0 = (2n-1) - \beta - \gamma n - \delta n^2$$

$$0 = (-1 - \beta) - n(2 + \gamma) - (2 - \delta) n^2$$

So $\beta = 1, \gamma = -2, \delta = 2$

$$R_n = A(n) + B(n)\beta + C(n)\gamma + D(n)\delta$$

Substituting $\alpha = 0, \beta = 1, \gamma = -2, \delta = 2$

$$(-1)^n n^2 = 0 + B(n)(1) + C(n)(-2) + D(n)(2)$$

$$(-1)^n n^2 = -(-B(n) + 2C(n)) + 2D(n)$$

From equation (4) $\{(-1)^n n = -B(n) + 2C(n)\}$

$$(-1)^n n^2 = -(-1)^n n + 2D(n)$$

$$2D(n) = (-1)^n (n^2 + n)$$

$$\mathbf{D(n) = ((-1)^n (n^2 + n))/2}$$

(for all $n \in N$) ……………………**(6)**

Closed form using equation (1),(2),(3),(5),(6)

$$R_n = \alpha + ((\beta (-1)^{n)} / 2) + ((\gamma (-1)^n (2n + 1))/ 4)$$

$$+ ((\delta (-1)^n (n^2 + n)) / 2)$$

$$S_n = \sum_{k=0}^{n} (-1)^k k^2 \qquad \text{(for all } n \in \mathbb{N})$$

$$S_0 = 0$$

$$S_n = S_{n-1} + (-1)^n n^2$$

A special case of

$R_0 = \alpha$

$R_n = R_{n-1} + (-1)^n (\beta + \gamma n + \delta n^2)$

A Special case of with : $\alpha = 0$, $\beta = 0$, $\gamma = 0$, $\delta = 1$

With Closed form

$$R_n = \alpha + ((\beta (-1)^n)/2) + ((\gamma (-1)^n (2n+1))/4) + ((\delta (-1)^n (n^2 + n))/2)$$

Substituting $\alpha = 0$, $\beta = 0$, $\gamma = 0$, $\delta = 1$

$$R_n = S_n = ((-1)^n (n^2 + n)) / 2$$

$$\sum_{k=0}^{n} (-1)^2 k^2 = ((-1)^n (n^2 + n)) / 2$$

Problem 14

Evaluate $\sum_{k=1}^{n} k2^k$

by rewriting it as multiple sum $\sum_{1 \le j \le k \le n} 2^k$

$$k = \sum_{j=1}^{k} 1$$

$$\sum_{k=1}^{n} k2^k \quad = \sum_{k=1}^{n}\sum_{j=1}^{n}(1) * 2^k$$

$$\sum_{k=1}^{n}\sum_{j=1}^{k} 2^k$$

The multiple sum can be solved in 2 ways:

1. Take sum over j first

But it gives the original sum

2. Take sum over k first

(1 ≤ j ≤ k) and (1 ≤ k ≤ n) ≡ (1 ≤ j ≤ k ≤ n)

The geometric series formula is given as:

$$\sum_{j=k}^{n} r^j = \frac{r^{n+1} - r^k}{r - 1}$$

$$= \sum_{j=1}^{n} \frac{2^{n+1} - 2^{j}}{2 - 1} \quad \text{(geometric series)}$$

$$= \sum_{j=1}^{n} [2^{n+1} - 2^{j}]$$

$$= \sum_{j=1}^{n} [2^{n+1}] - \sum_{j=1}^{n} [2^{j}] \quad \text{(distributive law)}$$

$$\sum_{j=1}^{n}\sum_{k=j}^{n} 2^k = 2^{n+1}\sum_{j=1}^{n} 1 - \sum_{j=1}^{n} [2^j]$$

$$= 2^{n+1} \ (n) - \ [2^{n+1} - \ 2 \ ] \ / \ (2\text{-}1)$$

(geometric series formula)

$$= 2^{n+1} \ (n) - \ [\ 2^{n+1} - \ 2 \ ]$$

$$= 2^{n+1} \ (n - 1) + 2$$

The final solution is:

$$\sum_{k=1}^{n} k2^k = 2^{n+1}(n-1) + 2$$