Chapter 2

Solutions to Homework Problems

Contents

This notes includes selected exercise problems from second chapter of Concrete Mathematics ([CM]) by Graham, Knuth, and Patashnik.

If you find any mistakes in the notes, please feel free to correct it.

1 Problem 5

Whats wrong with the following derivation?

$(\sum_{j=1}^{n} a_j)(\sum_{k=1}^{n} \frac{1}{a_k}) = \sum_{j=1}^{n} \sum_{k=1}^{n} \frac{a_j}{a_k} = \sum_{j=1}^{n} \sum_{k=1}^{n} \frac{a_k}{a_k} = \sum_{k=1}^{n} n = n^2$

1.1 Solution 5

Correctness Verification

We want to see whether the derivation is correct or not

For this we set n = 3 and we want to see if the right part of the derivation is equal to the left part

$S_L = (\sum_{j=1}^{3} a_j)(\sum_{k=1}^{3} \frac{1}{a_k})$

$S_L = (\sum_{j=1}^{3} a_j)(\frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{a_3})$

$S_L = (a_1 + a_2 + a_3)(\frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{a_3})$

$S_L = a_1(\frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{a_3}) + a_2(\frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{a_3}) + a_3(\frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{a_3})$

$S_L = 1 + \frac{a_1}{a_2} + \frac{a_1}{a_3} + \frac{a_2}{a_1} + 1 + \frac{a_2}{a_3} + \frac{a_3}{a_1} + \frac{a_3}{a_2} + 1$

$S_L = 3 + \frac{a_2+a_3}{a_1} + \frac{a_1+a_3}{a_2} + \frac{a_1+a_2}{a_3}$

$S_R = \sum_{j=1}^{3} \sum_{k=1}^{3} \frac{a_k}{a_k}$

$S_R = \sum_{j=1}^{3} (\frac{a_1}{a_1} + \frac{a_2}{a_2} + \frac{a_3}{a_3})$

$S_R = \sum_{j=1}^{3} 3$

$S_R = 9 = 3^2$

We can see that $S_L != S_R$, so we detect that the derivation is not correct

How can we find the error?

Idea: check every step of the derivation

We have 2 derivation steps (see below):

$$(\sum_{j=1}^{n} a_j)(\sum_{k=1}^{n} \frac{1}{a_k}) = \sum_{j=1}^{n} \sum_{k=1}^{n} \frac{a_j}{a_k} \tag{1}$$

$\sum_{j=1}^{n}\sum_{k=1}^{n}\frac{a_j}{a_k}=\sum_{j=1}^{n}\sum_{k=1}^{n}\frac{a_k}{a_k}=\sum_{k=1}^{n}n=n^2 \quad (2)$

We want to check which one is wrong

Derivation step 1
We check the first step of the derivation

$(\sum_{j=1}^{n}a_j)(\sum_{k=1}^{n}\frac{1}{a_k})=\sum_{j=1}^{n}\sum_{k=1}^{n}\frac{a_j}{a_k} \quad (1)$

Can we do this step? Yes
Why? Based on the General Distributive Law

Derivation step 2
We check the first step of the derivation

$\sum_{j=1}^{n}\sum_{k=1}^{n}\frac{a_j}{a_k}=\sum_{j=1}^{n}\sum_{k=1}^{n}\frac{a_k}{a_k} \quad (2)$

Can we do this step? No!
Why? Because the transformation is not in accordance with the changing of the indexes in multiple sums rule.

We check the first step of the derivation
From logic we know that in the multiple sum S in this step, k is a bound variable to the inner sum, while j is a bound variable to the exterior sum.
But in the multiple sum S, k is a bound variable both to the inner sum, and to the exterior sum.
Based on the substitution rules of predicate logic, we cannot substitute j of the outer sum with the same k as the one in the inner sum.
The substitution works only when

$a_j - a_k, \forall i, j, 1 \leqslant j, k \leqslant n$

Why? Because then we will have:

$S_L = n + (\frac{a_2+a_3+...+a_n}{a_1}) + (\frac{a_1+a_3++a_4+...+a_n}{a_2}) + ... + (\frac{a_1+...+a_{k-1}+a_k+...+a_n}{a_k}) + \cdots + (\frac{a_1+a_2+...+a_{n-1}}{a_n})$

$S_L = n + \frac{(n-1)a_1}{a_1} + ... + \frac{(n-1)a_n}{a_n}$

$S_L = n + (n-1)n = n^2 = S_R$

2 Problem 6

What is the value of $\sum_k [1 \le j \le k \le n]$, as a function of j and n?

2.1 Solution 6

We start simplifying the expression,

$$\begin{aligned}
&\sum_k [1 \le j \le k \le n] \\
&= \sum_{1 \le j \le k \le n} 1 \qquad \left(\because \sum_k [P(k)] = \sum_{P(k)} 1 \right) \\
&= \sum_{1 \le j \le n} \left(\sum_{j \le k \le n} 1 \right) \qquad (\because (1 \le j \le k \le n) = (1 \le j \le n) \cap (j \le k \le n)) \\
&= \sum_{1 \le j \le n} (n - j + 1)
\end{aligned}$$

3 Problem 7

Let $\nabla(f(x)) = f(x) - f(x-1)$. What is $\nabla(x^{\overline{m}})$?

3.1 Solution 7

We define rising factorial power, $x^{\overline{m}}$, as, $x^{\overline{m}} = x(x+1)(x+2)\cdots(x+m-1), m > 0$.

We want to evaluate, $\nabla(x^{\overline{m}}) = x^{\overline{m}} - (x-1)^{\overline{m}}$

This can be simply done by putting the values for x and x-1 in the equation. Now,

$$x^{\overline{m}} = x(x+1)(x+2)\cdots(x+m-1), m > 0.$$

$$\begin{aligned}(x-1)^{\overline{m}} &= (x-1)x(x+1)\cdots(x-1+m-1), m > 0.\\ &= (x-1)x(x+1)\cdots(x+m-2)\end{aligned}$$

$$\begin{aligned}\nabla(x^{\overline{m}}) &= x^{\overline{m}} - (x-1)^{\overline{m}}\\ &= x(x+1)(x+2)\cdots(x+m-1) - (x-1)x(x+1)\cdots(x+m-2)\\ &= (x-1)x(x+1)\cdots(x+m-2)(x+m-1-x+1)\\ &= m(x-1)x(x+1)\cdots(x+m-2)\\ \\ &= mx^{\overline{m-1}}\end{aligned}$$

$$\boxed{\nabla(x^{\overline{m}}) = mx^{\overline{m-1}}}$$

A point to note that $\nabla(x^{\overline{m}})$ is not equal to $\triangle(x^{\overline{m}})$, where $\triangle(f(x)) = f(x+1) - f(x)$.

$$\begin{aligned}\triangle(x^{\overline{m}}) &= (x+1)^{\overline{m}} - x^{\overline{m}}\\ &= (x+1)(x+2)\cdots(x+m) - x(x+1)\cdots(x+m-1)\\ &= x(x+1)\cdots(x+m-1)(x+m-x)\\ &= mx(x+1)\cdots(x+m-1)\\ \\ &= m(x+1)^{\overline{m-1}}\end{aligned}$$

$$\boxed{\triangle(x^{\overline{m}}) = m(x+1)^{\overline{m-1}}}$$

Thus what we learn from this exercise is, $\boxed{\nabla(x^{\overline{m}}) = mx^{\overline{m-1}} \neq \triangle(x^{\overline{m}}) = m(x+1)^{\overline{m-1}}}$

4 Problem 8

What is the value of $0^{\underline{m}}$, when m is a given integer?

4.1 Solution 8

Definition of $x^{\underline{m}}$ and $x^{\underline{-m}}$

$x^{\underline{m}} = x(x-1)...(x-m+1)$

From: (2.43) Concrete MathematicsA Foundation for Computer ScienceGraham, Knuth, Patashnik

$x^{\underline{-m}} = \frac{1}{(x+1)(x+2)...(x+m)}$

For $m \geqslant 1$

For $m \geqslant 1$ we use the definition $x^{\underline{m}} = x(x-1)...(x-m+1)$
x = 0 will always give us a product of 0.
0 = 0(0-1)(0-m+1)

For $m \leqslant 0$

For m $\geqslant$ 1 we use the definition $x^{\underline{-m}} = \frac{1}{(x+1)(x+2)...(x+m)}$

$0^{\underline{-m}} = \frac{1}{((0+1)(0+2)...(0+|m|))}$

$= \frac{1}{((1)(2)...(|m|))}$

$= \frac{1}{(|m|!)}$

Conclusion.

What is the value of $0^{\underline{-m}}$,
when m is a given integer?

0, if m$\geqslant$1;

$\frac{1}{(|m|!)}$, if m $\leqslant$ 0.

5 Problem 10

The text derives the following formula for the difference of a product.

$$\Delta(uv) = u\Delta v - E_v\Delta u$$

How can this formula be correct, when the left hand side is symmetric with respect to u and v but the right side is not?

5.1 Solution 10

Let us derive the formula for $\Delta(uv)$ in all possible ways.

Derivation 1

$$\begin{aligned}
&\Delta(uv) \\
&= u(x+1)v(x+1) - u(x)v(x) \\
&= u(x+1)v(x+1) - u(x)v(x) - u(x)v(x+1) + u(x)v(x+1) \\
&= u(x)(v(x+1) - v(x)) - v(x+1)(u(x+1) - u(x)) \\
&= u(x)\Delta v - v(x+1)\Delta u \\
&= u\Delta v - E_v\Delta u \qquad (E_v = v(x+1))
\end{aligned}$$

Derivation 2

$$\begin{aligned}
&\Delta(uv) \\
&= u(x+1)v(x+1) - u(x)v(x) \\
&= u(x+1)v(x+1) - u(x)v(x) - u(x+1)v(x) + u(x+1)v(x) \\
&= v(x)(u(x+1) - u(x)) - u(x+1)(v(x+1) - v(x)) \\
&= v(x)\Delta u - u(x+1)\Delta v \\
&= v\Delta u - E_u\Delta v \qquad (E_u = u(x+1))
\end{aligned}$$

We see that

$$\begin{aligned}
\Delta(uv) &= u\Delta v - E_v\Delta u = v\Delta u - E_u\Delta v \\
\Delta(2uv) &= (u\Delta v + v\Delta u) - (E_u\Delta v + E_v\Delta u)
\end{aligned}$$

From the above arguments it is clear that if we just look at the equation

$$\Delta(uv) = u\Delta v - E_v\Delta u$$

it does not seem symmetric but if we see the complete equation

$$\Delta(uv) = u\Delta v - E_v\Delta u = v\Delta u - E_u\Delta v$$

it looks symmetric.

6 Problem 11

The general rule for summation by parts is equivalent to $\sum_{0\leqslant k<n}(a_{k+1}-a_k)b_k = a_nb_n - a_0b_0 - \sum_{0\leqslant k<n} a_{k+1}(b_{k+1}b_k)$, for n $\geqslant 0$

Prove this formula directly by using the distributive, associative, and commutative laws.

6.1 Solution 11

The general rule for summation by parts is equivalent to:

$\sum_{0\leqslant k<n}(a_{k+1}-a_k)b_k = a_nb_n - a_0b_0 - \sum_{0\leqslant k<n} a_{k+1}(b_{k+1}b_k)$, for n $\geqslant 0$

Prove this formula directly by using the
distributive, associative and commutative laws

$\sum_{0\leqslant k<n}(a_{k+1}-a_k)b_k$

$= \sum_{0\leqslant k<n}(a_{k+1}b_k - a_kb_k)$ (Distributive Law)

$= \sum_{k=0}^{n-1}(a_{k+1}b_k) - \sum_{k=0}^{n-1}(a_kb_k)$ (Associative Law)

We can write

$= \sum_{k=0}^{n-1}(a_kb_k) = \sum_{k=0}^{n}(a_kb_k) - a_nb_n$

$= \sum_{k=0}^{n-1}(a_{k+1}b_k) - \sum_{k=0}^{n}(a_kb_k) + a_nb_n$

$= \sum_{k=0}^{n-1} a_{k+1}b_k - \sum_{k=1}^{n} a_kb_k + a_nb_n - a_0b_0$

$= \sum_{k=0}^{n-1} a_{k+1}b_k - \sum_{k=0}^{n-1} a_{k+1}b_{k+1} + a_nb_n - a_0b_0$

$= \sum_{k=0}^{n-1}(a_{k+1}b_k - a_{k+1}b_{k+1}) + a_nb_n - a_0b_0$

$= \sum_{k=0}^{n-1} a_{k+1}(b_k - b_{k+1}) + a_nb_n - a_0b_0$

$= a_nb_n - a_0b_0 - \sum_{k=0}^{n-1} a_{k+1}(b_{k+1} - b_k)$

So we have proved

$\sum_{0 \leq k \leq n}(a_{k+1} - a_k)b_k = a_n b_n - a_0 b_0 - \sum_{0 \leq k \leq n} a_{k+1}(b_{k+1} - b_k)$, for n ≥ 0

7 Problem 13

Use the repertoire method to find a closed form for $\sum_{k=0}^{n}(-1)^n k^2$

7.1 Solution 13

We wish to find the sum S_n, given by, $S_n = \sum_{k=0}^{n}(-1)^n k^2$. Now putting the sum into a recursive form as follows we get,

$$\boxed{\begin{aligned} &S_0 = 0 \\ &S_n = S_{n-1} + (-1)^n n^2 \end{aligned}}$$

We represent the most general form for the recursive expression shown above as follows:

$$\boxed{\begin{aligned} &R_0 = \alpha \\ &R_n = R_{n-1} + (-1)^n(\beta + n\gamma + n^2\triangle) \end{aligned}}$$

The actual values in the above generalization $\alpha = 0, \beta = 0, \gamma = 0, \triangle = 1$. To get a solution in the closed form, we express R_n in a generalized form. We write R_n as:

$$\boxed{R_n = A(n)\alpha + B(n)\beta + C(n)\gamma + D(n)\triangle}$$

Strategy:

§ We need to solve this equation to find the values of A(n), B(n), C(n) and D(n).

§ We will pick simple functions (R_n) with easy values for $\alpha, \beta, \gamma, \triangle$ and the solve the equation to find value of A(n), B(n), C(n) and D(n).

§ Substitute the values of A(n), B(n), C(n) and D(n) to get the general equation for the recurrence.

Case 1: Taking $R_n = 1$, for all n ε N
$R_0 = \alpha$, $R_0 = 1$. Thus we get, $\alpha = 1$

$$\boxed{\begin{aligned} &R_n = R_{n-1} + (-1)^n(\beta + n\gamma + n^2\triangle) \\ &or, 1 = 1 + (-1)^n(\beta + n\gamma + n^2\triangle) \\ &or, 0 = (-1)^n(\beta + n\gamma + n^2\triangle) \\ &or, (\beta + n\gamma + n^2\triangle) = 0 \\ &or, (\beta - 0) + n(\gamma - 0) + n^2(\triangle - 0) = 0 \end{aligned}}$$

Putting the values, in $R_n = A(n)\alpha + B(n)\beta + C(n)\gamma + D(n)\triangle$, we have, $R_n = A(n).1 + 0$. Since $R_n = 1$, $\boxed{\mathbf{A(n) = 1}}$, $(\forall n \varepsilon N)$

Case 2: Taking $R_n = (-1)^n$, for all n ε N
$R_0 = \alpha$, $R_0 = (-1)^0$. Thus we get, $\alpha = 1$

$$\boxed{\begin{aligned} &R_n = R_{n-1} + (-1)^n(\beta + n\gamma + n^2\triangle) \\ &or, (-1)^n = (-1)^{n-1} + (-1)^n(\beta + n\gamma + n^2\triangle) \\ &or, 1 = -1 + 1.(\beta + n\gamma + n^2\triangle) \\ &or, (\beta + n\gamma + n^2\triangle) = 2 \\ &or, (\beta - 2) + n(\gamma - 0) + n^2(\triangle - 0) = 0 \end{aligned}}$$

So we have $\beta = 2, \gamma = 0, \triangle = 0$. Putting the values, in $R_n = A(n)\alpha + B(n)\beta + C(n)\gamma + D(n)\triangle$, we have,
$R_n = A(n).1 + B(n).2 + C(n).0 + D(n).0 = (-1)^n$

$$\boxed{B(n) = \frac{(-1)^n - 1}{2}}$$

Case 3: Taking $R_n = (-1)^n n$, for all n ε N
$R_0 = \alpha$, $R_0 = (-1)^0.0 = 0$. Thus we get, $\alpha = 0$

$$\boxed{\begin{aligned} &R_n = R_{n-1} + (-1)^n(\beta + n\gamma + n^2\triangle) \\ &or, n(-1)^n = (n-1)(-1)^{n-1} + (-1)^n(\beta + n\gamma + n^2\triangle) \\ &or, 0 = (2n - 1) + (-1).(\beta + n\gamma + n^2\triangle) \\ &or, (\beta + n\gamma + n^2\triangle) = 2n - 1 \\ &or, (\beta + 1) + n(\gamma - 2) + n^2(\triangle - 0) = 0 \end{aligned}}$$

So we have $\beta = -1, \gamma = 2, \triangle = 0$. Putting the values, in $R_n = A(n)\alpha + B(n)\beta + C(n)\gamma + D(n)\triangle$, we have,
$R_n = A(n).0 - B(n).1 + C(n).2 + D(n).0 = n(-1)^n$
$or, -B(n) + 2C(n) = n(-1)^n$

$$\boxed{C(n) = \frac{((-1)^n(2n+1) - 1)}{4}}$$

Case 4: Taking $R_n = (-1)^n n^2$, for all n ε N
$R_0 = \alpha$, $R_0 = (-1)^0.0^2 = 0$. Thus we get, $\alpha = 0$

$$\boxed{\begin{aligned}&R_n = R_{n-1} + (-1)^n(\beta + n\gamma + n^2\triangle)\\ &or, n^2(-1)^n = (n-1)^2(-1)^{n-1} + (-1)^n(\beta + n\gamma + n^2\triangle)\\ &or, 0 = (2n^2 - 2n + 1) + (-1).(\beta + n\gamma + n^2\triangle)\\ &or, (\beta + n\gamma + n^2\triangle) = 2n^2 - 2n + 1\\ &or, (\beta - 1) + n(\gamma + 2) + n^2(\triangle - 2) = 0\end{aligned}}$$

So we have $\beta = 1, \gamma = -2, \triangle = 2$. Putting the values, in $R_n = A(n)\alpha + B(n)\beta + C(n)\gamma + D(n)\triangle$, we have,

$R_n = A(n).0 + B(n).1 - C(n).2 + D(n).2 = n^2(-1)^n$

$or, n^2(-1)^n = -(-B(n) + 2C(n)) + 2D(n)$

$or, n^2(-1)^n = -(n(-1)^n) + 2D(n)$

$$\boxed{D(n) = \frac{((-1)^n(n+n^2))}{2}}$$

Thus putting the derived values for A(n), B(n), C(n) and D(n) in R_n, we get,

$$\boxed{R_n = \alpha + \beta\frac{(-1)^n - 1}{2} + \gamma\frac{((-1)^n(2n+1) - 1)}{4} + \triangle\frac{((-1)^n(n+n^2))}{2}}$$

Now for the given sum, $S_n = \sum_{k=0}^{n}(-1)^n k^2$, it is a special case with, $\alpha = 0, \beta = 0, \gamma = 0$ and $\triangle = 1$. Putting this respective values in the final equation for R_n, we get,

$R_n = S_n = \frac{((-1)^n(n+n^2))}{2}$.

Thus,

$$\boxed{\sum_{k=0}^{n}(-1)^n k^2 = \frac{((-1)^n(n+n^2))}{2}}$$

8 Problem 14

Evaluate $\sum_{k=1}^{n} k2^k$ by rewriting it as the multiple sum $\sum_{1\le j\le k\le n} 2^k$.

8.1 Solution 14

$$
\begin{aligned}
&\sum_{k=1}^{n} k2^k \\
&= \sum_{k=1}^{n} \left(\sum_{j=1}^{k} 1 \right) 2^k \\
&= \sum_{k=1}^{n} \sum_{j=1}^{k} 2^k \\
&= \sum_{1\le j\le k\le n} 2^k \\
&= \sum_{j=1}^{n} \sum_{k=j}^{n} 2^k \\
&= \sum_{j=1}^{n} \left(\sum_{k=0}^{n} 2^k - \sum_{k=0}^{j-1} 2^k \right) \\
&= \sum_{j=1}^{n} \left(2^{n+1} - 2^j \right) \\
&= 2^{n+1} \sum_{j=1}^{n} 1 - \sum_{j=1}^{n} 2^j \\
&= 2^{n+1}(n) - (2^{n+1} - 2) \\
&= 2^{n+1}(n-1) + 2
\end{aligned}
$$

9 Problem 15

Evaluate $⧉_n = \sum_{k=1}^{n} k^3$

9.1 Solution 15

Note

Calculate sum of the first 10 cubes

n = 1,2,3,4,5,6,7,8,9,10,..

n^3 = 1,8,27,64,125,216,343,512,729,1000,..

$⧉_n$ = 1,9,36,100,225,441,784,1296,2025,3025,..

As evident we cannot find a closed form for $⧉_n$, directly.

Review Method 5: Expand and Contract

Finding a closed form for the sum of the first n squares,

$\square_n = \sum_{k=1}^{n} k^2$ for n ≥ 0

$= \sum_{0 \leq j \leq k \leq n} k = \sum_{0 \leq j \leq n} \sum_{j \leq k \leq n} k$

Since Average = (1st + last) / 2

$= \sum_{0 \leq j \leq n} (\frac{j+n}{2})(n-j+1)$

$= \frac{1}{2} \sum_{0 \leq j \leq n} (n(n+1) + j - j^2)$

$= \frac{1}{2} n^2 (n+1) + \frac{1}{4} n(n+1) - \frac{1}{2} \sum_{0 \leq j \leq n} j^2$

$= \frac{1}{2} n^2 (n+1) + \frac{1}{4} n(n+1) - \frac{1}{2} \quad \square_n$

$= \frac{1}{2} n(n + \frac{1}{2})(n+1) - \frac{1}{2} \quad \square_n$

Upper Triangle Sum

Consider the array of n^2 products $a_j\ a_k$
Our goal will be to find a simple formula for the sum of all elements on or above the diagonal of this array.
Because $a_j a_k = a_k a_j$, the array is symmetrical about its main diagonal; therefore will be approximately half the sum of all the elements.
$S_{\lhd} = \sum_{0 \le j \le k \le n} a_j a_k = \sum_{0 \le k \le j \le n} a_k a_j = \sum_{0 \le k \le j \le n} a_j a_k = S_{\rhd}$

because we can rename (j,k) as (k,j) , furthermore, since

$(1 \le j \le k \le n) \cap (1 \le k \le j \le n) \equiv (1 \le j, k \le n) \cap (1 \le k = j \le n)$

we have $2S_{\lhd} = S_{\lhd} + S_{\lhd} = \sum_{0 \le j \le k \le n} a_j a_k + \sum_{0 \le k \le j \le n} a_j a_k = \sum_{0 \le j, k \le n} a_j a_k + \sum_{0 \le j = k \le n} a_j a_k$

The first sum is

$(\sum_{j=1}^{n} a_j)(\sum_{k=1}^{n} a_k) = (\sum_{k=1}^{n} a_k)^2$

by the general distributive law

the second sum is

$(\sum_{k=1}^{n} a_k^2)$

$S_{\lhd} = \sum_{0 \le j \le k \le n} a_j a_k = \frac{1}{2}((\sum_{k=1}^{n} a_k)^2 + (\sum_{k=1}^{n} a_k^2))$

An expression for the upper triangular sum in terms of simpler single sums

$\boxed{\square}_n + \quad \square_n = \sum_{k=1}^{n} k^3 + \sum_{k=1}^{n} k^2$

$= \sum_{k=1}^{n} (k^3 + k^2)$ [associative law]

$= \sum_{k=1}^{n} k^2 (k+1)$

$= \sum_{k=1}^{n} k * k * (k+1)$

$= \sum_{k=1}^{n} 2 * \frac{1}{2} * k * k * (k+1)$

$= 2 \sum_{k=1}^{n} k * \frac{1}{2} * k * (k+1)$ [distributive law]

Now,

$\frac{1}{2} * k * (k+1) = \sum_{j=1}^{k} j$ (sum of first k natural numbers)

$= 2 \sum_{k=1}^{n} k * \frac{1}{2} * k * (k+1) = 2(\sum_{k=1}^{n} k)(\sum_{j=1}^{k} j)$

Now we would use general distributive law that states

$\sum_{j \in J, k \in K} a_j a_k = (\sum_{j \in J} a_j)(\sum_{k \in K} b_k)$

$\sum_{k=1}^{n} k \sum_{j=1}^{k} j = \sum_{k=1}^{n} \sum_{j=1}^{k} k * j = \sum_{j=1}^{k} \sum_{k=1}^{n} k * j$ (as k is a constant with respect to j)

$since, (1 \leq k \leq n) \cap (1 \leq j \leq k) \equiv (1 \leq j \leq k \leq n)$

$therefore, \sum_{j=1}^{k} \sum_{k=1}^{n} k * j = \sum_{1 \leq j \leq k \leq n} k * j$

So, our original equation becomes -

$2 \sum_{k=1}^{n} k * \frac{1}{2} * k * (k+1) = 2(\sum_{k=1}^{n} k)(\sum_{j=1}^{k} j)$

$= 2 \sum_{1 \leq j \leq k \leq n} k * j$

we proved

$\text{⧉}_n + \quad \square_n = 2 \sum_{1 \leq j \leq k \leq n} jk$

We use the expression for upper triangular sum for further evaluation as shown below

$\text{⧉}_n + \quad \square_n = 2 * \frac{1}{2}((\sum_{k=1}^{n} k)^2 + \sum_{k=1}^{n} k^2)$

$= (\sum_{k=1}^{n} k)^2 + \sum_{k=1}^{n} k^2$

The first term is the square of the summation of the first n natural numbers and the second term is the sum of the first n squares, i.e., $\square_n$

$$\cube_n + \quad \square_n = (\frac{n(n+1)}{2})^2 + \quad \square_n$$

Therefore $\cube_n = (\frac{n(n+1)}{2})^2$

10 Problem 16

Prove that $\frac{x^{\underline{m}}}{(x-n)^{\underline{m}}} = \frac{x^{\underline{n}}}{(x-m)^{\underline{n}}}$, unless one of the denominators is 0.

10.1 Solution 16

Let's start by a quick revision of the definition of $x^{\underline{n}}$.

When $n > 0$,

$\boxed{x^{\underline{n}} = x(x-1)(x-2)\cdots(x-n+1) = \frac{x!}{(x-n)!}}$

When $n < 0$,

$\boxed{x^{\underline{n}} = \frac{1}{(x+1)(x+2)\cdots(x-n)}}$

When $n = 0$,

$\boxed{x^{\underline{n}} = 1}$

♣ **Case 1: n = 0 and m = 0**

Trivial!

♣ **Case 2:** $n > 0$ **and** $m > 0$

We will use the definition, $x^{\underline{n}} = \frac{x!}{(x-n)!}$, for n greater than 0.

$$\frac{x^{\underline{m}}}{(x-n)^{\underline{m}}} = \frac{\frac{x!}{(x-m)!}}{\frac{(x-n)!}{(x-n-m)!}}$$

$$= \frac{x!(x-n-m)!}{(x-n)!(x-m)!}$$

$$\frac{x^{\underline{n}}}{(x-m)^{\underline{n}}} = \frac{\frac{x!}{(x-n)!}}{\frac{(x-m)!}{(x-n-m)!}}$$

$$= \frac{x!(x-n-m)!}{(x-n)!(x-m)!}$$

So,

$$\frac{x^{\underline{m}}}{(x-n)^{\underline{m}}} = \frac{x^{\underline{n}}}{(x-m)^{\underline{n}}}$$

♣ **Case 3:** $n < 0$ **and** $m < 0$

We will use the definition, $x^{\underline{n}} = \frac{1}{(x+1)\cdots(x-n)}$, for n lesser than 0.

$$\frac{x^{\underline{m}}}{(x-n)^{\underline{m}}} = \frac{\frac{1}{(x+1))\cdots(x-m)}}{\frac{1}{(x-n+1)\cdots(x-n-m)}}$$

$$= \frac{(x-n+1)\cdots(x-n-m)}{(x+1)\cdots(x-m)}$$

$$\frac{x^{\underline{n}}}{(x-m)^{\underline{n}}} = \frac{\frac{1}{(x+1)\cdots(x-n)}}{\frac{1}{(x-m+1)\cdots(x-n-m)}}$$

$$= \frac{(x-m+1)\cdots(x-n-m)}{(x+1)\cdots(x-n)}$$

Without loss of generality, we will assume, $n \leq m < 0$, so that, $0 < -m \leq -n$

$$\frac{x^{\underline{n}}}{(x-m)^{\underline{n}}} = \frac{(x-m+1)\cdots(x-n-m)}{(x+1)\cdots(x-n)}$$

$$= \frac{(x-m+1)\cdots x(x-1)\cdots(x-m+1)}{(x+1)\cdots(x-m)(x-m+1)\cdots(x-n)}$$

$$= \frac{x^{\underline{m}}}{(x-n)^{\underline{m}}}$$

♣ **Case 4:** $n < 0$ **and** $m > 0$
Sub-Case a: If $-m-n \geq 1$**,**

$$\frac{x^{\underline{m}}}{(x-n)^{\underline{m}}} = \frac{x(x-1)\cdots(x-m+1)}{(x-n)(x-n-1)\cdots(x-n-m+1)}$$

$$\frac{x^{\underline{n}}}{(x-m)^{\underline{n}}} = \frac{\frac{1}{(x+1)\cdots(x-n)}}{\frac{1}{(x-m+1)\cdots(x-n-m)}}$$

$$= \frac{(x-m+1)\cdots(x-n-m)}{(x+1)\cdots(x-n)}$$

$$= \frac{(x-m+1)\cdots x(x+1)\cdots(x-n-m)}{(x+1)\cdots(x-m-n)(x-m-n+1)\cdots(x-n)}$$

$$= \frac{x^{\underline{m}}}{(x-n)^{\underline{m}}}$$

Sub-Case b: If $-m-n \leq -1$**,**

$$\frac{x^{\underline{n}}}{(x-m)^{\underline{n}}} = \frac{(x-m+1)\cdots(x-n-m)}{(x+1)\cdots(x-n)}$$

$$\frac{x^{\underline{m}}}{(x-n)^{\underline{m}}} = \frac{x(x-1)\cdots(x-m+1)}{(x-n)(x-n-1)\cdots(x-n-m+1)}$$

$$= \frac{x(x-1)\cdots(x-n-m+1)(x-n-m)\cdots(x-m+1)}{(x-n)(x-n-1)\cdots(x+1)x\cdots(x-n-m+1)}$$

$$= \frac{(x-m+1)\cdots(x-m-n)}{(x+1)\cdots(x-n)}$$

$$= \frac{x^{\underline{n}}}{(x-m)^{\underline{n}}}$$

Sub-Case c: If $-1 < -m-n < 1, \Longrightarrow -m-n = 0$**,**

$$\frac{x^{\underline{n}}}{(x-m)^{\underline{n}}} = \frac{(x-m+1)\cdots(x-n-m)}{(x+1)\cdots(x-n)}$$

$$= \frac{(x-m+1)\cdots x}{(x-n-m+1)\cdots(x-n)}$$

$$= \frac{x^{\underline{m}}}{(x-n)^{\underline{m}}}$$

♣ **Case 5:** $n > 0$ **and** $m < 0$

This is exactly symmetrical to Case 4. Just swap m and n.

11 Problem 17

Show that the following formulas can be used to convert between rising and falling factorial powers, for all integers m

a) $x^{\overline{m}} = (-1)^m(-x)^{\underline{m}} = (x+m-1)^{\underline{m}} = 1/(x-1)^{\underline{-m}}$

b) $x^{\underline{m}} = (-1)^m(-x)^{\overline{m}} = (x-m+1)^{\overline{m}} = 1/(x+1)^{\overline{-m}}$

The answer to exercise 9 defines $x^{\overline{-m}}$

11.1 Solution 17

The formulas for $x^{\overline{m}}$ and $x^{\underline{m}}$ are

$$x^{\overline{m}} = \begin{cases} x(x+1)\ldots(x+m-1) & m > 0 \\ \frac{1}{(x-1)(x-2)\ldots(x+m)} & m < 0 \\ 1 & m = 0 \end{cases}$$

$$x^{\underline{m}} = \begin{cases} x(x-1)\ldots(x-m+1) & m > 0 \\ \frac{1}{(x+1)(x+2)\ldots(x-m)} & m < 0 \\ 1 & m = 0 \end{cases}$$

Case 1: m = 0

We know that $x^{\overline{0}} = 1$. Hence, when $m = 0$ all the terms below will be equal to 1.

a) $x^{\overline{m}} = (-1)^m(-x)^{\underline{m}} = (x+m-1)^{\underline{m}} = 1/(x-1)^{\underline{-m}}$

b) $x^{\underline{m}} = (-1)^m(-x)^{\overline{m}} = (x-m+1)^{\overline{m}} = 1/(x+1)^{\overline{-m}}$

Hence, the equations are true for $m = 0$

Case 2: m > 0

a)

$$x^{\overline{m}} = x(x+1)\ldots(x+m-1)$$

$$\begin{aligned} (-1)^m(-x)^{\underline{m}} &= (-1)^m(-x)(-x-1)\ldots(-x-m+1) \\ &= x(x+1)\ldots(x+m-1) \end{aligned}$$

$$\begin{aligned} (x+m-1)^{\underline{m}} &= (x+m-1)\ldots(x+1)x \\ &= x(x+1)\ldots(x+m-1) \end{aligned}$$

$$\begin{aligned}1/(x-1)^{\underline{-m}} &= (x-1+1)(x-1+2)\dots(x-1+m)\\ &= x(x+1)\dots(x+m-1)\end{aligned}$$

Hence, the equation is true

b)

$$x^{\underline{m}} = x(x-1)\dots(x-m+1)$$

$$\begin{aligned}(-1)^m(-x)^{\overline{m}} &= (-1)^m(-x)(-x+1)\dots(-x+m-1)\\ &= x(x-1)\dots(x-m+1)\end{aligned}$$

$$\begin{aligned}(x-m+1)^{\overline{m}} &= (x-m+1)\dots(x-1)x\\ &= x(x-1)\dots(x-m+1)\end{aligned}$$

$$\begin{aligned}1/(x+1)^{\overline{-m}} &= (x+1-1)(x+1-2)\dots(x+1-m)\\ &= x(x-1)\dots(x-m+1)\end{aligned}$$

Hence, the equation is true

Case 3: m < 0

a)

$$x^{\overline{m}} = \frac{1}{(x-1)(x-2)\dots(x+m)}$$

$$\begin{aligned}(-1)^m(-x)^{\underline{m}} &= \frac{(-1)^m}{(-x+1)(-x+2)\dots(-x-m)}\\ &= \frac{1}{(x-1)(x-2)\dots(x+m)}\end{aligned}$$

$$\begin{aligned}(x+m-1)^{\underline{m}} &= \frac{1}{(x+m-1+1)(x+m-1+2)\dots(x+m-1-m)}\\ &= \frac{1}{(x+m)(x+m+1)\dots(x-1)}\\ &= \frac{1}{(x-1)(x-2)\dots(x+m)}\end{aligned}$$

$$\begin{aligned}1/(x-1)^{\underline{-m}} &= \frac{1}{(x-1)(x-1-1)\dots(x-1+m-1)}\\ &= \frac{1}{(x-1)(x-2)\dots(x+m)}\end{aligned}$$

Hence, the equation is true

b)

$$x^{\underline{m}} = \frac{1}{(x+1)(x+2)\dots(x-m)}$$

$$\begin{aligned}(-1)^m(-x)^{\overline{m}} &= \frac{(-1)^m}{(-x-1)(-x-2)\dots(-x+m)} \\ &= \frac{1}{(x+1)(x+2)\dots(x-m)}\end{aligned}$$

$$\begin{aligned}(x-m+1)^{\overline{m}} &= \frac{1}{(x-m+1-1)(x-m+1-2)\dots(x-m+1+m)} \\ &= \frac{1}{(x-m)(x-m-1)\dots(x+1)} \\ &= \frac{1}{(x+1)(x+2)\dots(x-m)}\end{aligned}$$

$$\begin{aligned}1/(x+1)^{\overline{-m}} &= \frac{1}{(x+1)(x+1+1)\dots(x+1-m-1)} \\ &= \frac{1}{(x+1)(x+2)\dots(x-m)}\end{aligned}$$

Hence, the equation is true

12 Problem 19

Use a summation factor to solve the recurrence

$T_0 = 5$

$2T_n = nT_{n-1} + 3n!$

12.1 Solution 19

$T_1 = \frac{1.5+3.1}{2} = 4$

$T_2 = \frac{2.4+3.2.1}{2} = 7$

$T_3 = \frac{3.7+3.3.2.1}{2} = \frac{39}{2}$

$T_4 = \frac{4.\frac{39}{2}+3.4.3.2.1}{2} = 75$

$T_5 = \frac{5.75+3.5.4.3.2.1}{2} = \frac{735}{2}$

We can reduce the recurrence to a sum.

The general form is $a_nT_n = b_nT_{n-1} + c_n$

and comparing to our case $2T_n = nT_{n-1} + 3n!$

we can see that $c_n = 3n!$

By multiplying by a summation factor s_n on both sides of

$a_nT_n = b_nT_{n-1} + c_n$

we get

$s_na_nT_n = s_nb_nT_{n-1} + s_nc_n$

If we are able to impose

$s_n b_n = s_{n-1} a_{n-1}$

then we can rewrite the recurrence as

$S_n = S_{n-1} + s_n c_n$

where $S_n = s_n a_n T_n$

Expanding S_n we get

$S_n = s_1 b_1 T_0 + \sum_{k=1}^{n} s_k c_k$

and then the closed formula for T_n is

$T_n = \frac{1}{s_n a_n}(s_1 b_1 T_0 + \sum_{k=1}^{n} s_k c_k)$

By unfolding $s_n = s_{n-1}\frac{a_{n-1}}{b_n}$

, we obtain

$\frac{a_{n-1}...a1}{b_n...b_2}$

Since a_n = 2 and b_n = n

$s_n = \frac{a_{n-1}...a_1}{b_n...b_2} = \frac{2.2.2.2}{n(n-1)...2.1} = \frac{2^{n-1}}{n!}$

Remembering also that T_0 = 5 and c_n = 3 n!,

we can substitute in the closed formula for T_n

$T_n = \frac{1}{s_n a_n}(s_1 b_1 T_0 + \sum_{k=1}^{n} s_k c_k) = \frac{n!}{2^n}(5 + 3\sum_{k=1}^{n} 2^{k-1})$

$T_n = \frac{n!}{2^n}(5 + 3\sum_{k=1}^{n} 2^{k-1})$

$T_n = \frac{n!}{2^n}(5 + 3\sum_{1 \le k \le n} 2^{k-1})$

$T_n = \frac{n!}{2^n}(5 + 3\sum_{0 \leq k-1 \leq n-1} 2^{k-1})$

$T_n = \frac{n!}{2^n}(5 + 3\sum_{r=0}^{n-1} 2^r)$

where we set r = k - 1

We have seen that

$\sum_{k=0}^{n} x^k = \frac{x^{n+1}-1}{x-1}$, for $x \neq 1$

so in our case

$T_n = \frac{n!}{2^n}(5 + 3\sum_{r=0}^{n-1} 2^r)$

$= \frac{n!}{2^n}(5 + 3\frac{2^{(n-1)+1}-1}{2-1})$

$= \frac{n!}{2^n}(2 + 3.2^n)$

= n! $(2^{1-n} + 3)$

Checking the results

$T_0 = 5$

$2T_n = nT_{n-1} + 3n!$

$T_1 = \frac{1.5+3.1}{2} = 4$

$T_2 = \frac{2.4+3.2.1}{2} = 7$

$T_3 = \frac{3.7+3.3.2.1}{2} = \frac{39}{2}$

$T_4 = \frac{4.\frac{39}{2}+3.4.3.2.1}{2} = 75$

$T_5 = \frac{5.75+3.5.4.3.2.1}{2} = \frac{735}{2}$

Correct results

The solution of

$T_0 = 5$

$2T_n = nT_{n-1} + 3n!$ for n > 0

$T_n = n!(2^{1-n} + 3)$

13 Problem 20

Try to evaluate $\sum_{0\le k\le n} kH_k$ by perturbation, but deduce the value of $\sum_{0\le k\le n} H_k$ instead.

13.1 Solution 20

Let us look at the **Perturbation Method** first. Consider the sum, $S_n = \sum_{0\le k\le n} a_k$.

$$\begin{aligned} S_n + a_{n+1} = \sum_{0\le k\le n+1} a_k &= a_0 + \sum_{1\le k\le n+1} a_k \\ &= a_0 + \sum_{1\le k+1\le n+1} a_{k+1} \\ &= a_0 + \sum_{0\le k\le n} a_{k+1} \end{aligned}$$

§ **What is a Harmonic number (H_n)?** $\boxed{H_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} = \sum_{1\le k\le n} \frac{1}{k}}$

Now we calculate the required sum in the question as follows,

$$\begin{aligned} S_n + (n+1)H_{n+1} = \sum_{0\le k\le n+1} kH_k &= 0H_0 + \sum_{1\le k\le n+1} kH_k \\ &= 0 + \sum_{1\le k+1\le n+1} (k+1)H_{k+1} \\ &= 0 + \sum_{0\le k\le n} (k+1)H_{k+1} \end{aligned}$$

Now, $H_n = \sum_{1\le k\le n} \frac{1}{k}$. Therefore, $H_{n+1} = \sum_{1\le k\le n} \frac{1}{k} + \frac{1}{n+1}$.

$$
\begin{aligned}
S_n + (n+1)H_{n+1} &= \sum_{0\le k\le n} (k+1)H_{k+1} \\
&= \sum_{0\le k\le n} (k+1)(\frac{1}{k+1} + H_k) \\
&= \sum_{0\le k\le n} 1 + \sum_{0\le k\le n} kH_k + \sum_{0\le k\le n} H_k
\end{aligned}
$$

$$
\begin{aligned}
(n+1)H_{n+1} &= \sum_{0\le k\le n} 1 + \sum_{0\le k\le n} H_k \\
\sum_{0\le k\le n} H_k &= (n+1)H_{n+1} - \sum_{0\le k\le n} 1 \\
&= (n+1)H_{n+1} - (n+1)
\end{aligned}
$$

14 Problem 21

Evaluate the sums
a) $S_n = \sum_{k=0}^{n}(-1)^{n-k}$
b) $T_n = \sum_{k=0}^{n}(-1)^{n-k}k$
c) $U_n = \sum_{k=0}^{n}(-1)^{n-k}k^2$
by the perturbation method, assuming that $n \geq 0$

14.1 Solution 21

a) We consider

$$S_n = \sum_{k=0}^{n}(-1)^{n-k}$$

Split off the first term

$$\begin{aligned}
S_{n+1} &= \sum_{k=0}^{n+1}(-1)^{n+1-k} \\
&= (-1)^{n+1-0} + \sum_{k=1}^{n+1}(-1)^{n+1-k} \\
&= (-1)^{n+1} + \sum_{k+1=1}^{n+1}(-1)^{n+1-(k+1)} \\
&= (-1)^{n+1} + \sum_{k=0}^{n}(-1)^{n-k} \\
&= (-1)^{n+1} + S_n
\end{aligned}$$

Split off the last term

$$\begin{aligned}
S_{n+1} &= \sum_{k=0}^{n+1} (-1)^{n+1-k} \\
&= \sum_{k=0}^{n} (-1)^{n+1-k} + (-1)^{n+1-(n+1)} \\
&= \sum_{k=0}^{n} (-1)^{n+1-k} + 1 \\
&= -\sum_{k=0}^{n} (-1)^{n-k} + 1 \\
&= -S_n + 1
\end{aligned}$$

From the above two equations

$$\begin{aligned}
(-1)^{n+1} + S_n &= -S_n + 1 \\
\implies S_n &= \frac{1}{2}(1 - (-1)^{n+1}) \\
\implies S_n &= \frac{1}{2}(1 + (-1)^{n})
\end{aligned}$$

b) We consider

$$T_n = \sum_{k=0}^{n} (-1)^{n-k} k$$

Split off the first term

$$\begin{aligned}
T_{n+1} &= \sum_{k=0}^{n+1} (-1)^{n+1-k} k \\
&= (-1)^{n+1-0}.0 + \sum_{k=1}^{n+1} (-1)^{n+1-k} k \\
&= \sum_{k+1=1}^{n+1} (-1)^{n+1-(k+1)} (k+1) \\
&= \sum_{k=0}^{n} (-1)^{n-k} (k+1) \\
&= \sum_{k=0}^{n} (-1)^{n-k} k + \sum_{k=0}^{n} (-1)^{n-k} \\
&= T_n + S_n
\end{aligned}$$

Split off the last term

$$\begin{aligned}
T_{n+1} &= \sum_{k=0}^{n+1}(-1)^{n+1-k}k \\
&= \sum_{k=0}^{n}(-1)^{n+1-k}k + (-1)^{n+1-(n+1)}(n+1) \\
&= -\sum_{k=0}^{n}(-1)^{n-k}k + (n+1) \\
&= -T_n + (n+1) \\
&= (n+1) - T_n
\end{aligned}$$

From the above two equations

$$\begin{aligned}
& T_n + S_n = (n+1) - T_n \\
\implies & T_n = \frac{1}{2}(n+1-S_n) \\
\implies & T_n = \frac{1}{2}(n-(-1)^n)
\end{aligned}$$

c) We consider

$$U_n = \sum_{k=0}^{n}(-1)^{n-k}k^2$$

Split off the first term

$$\begin{aligned}
U_{n+1} &= \sum_{k=0}^{n+1}(-1)^{n+1-k}k^2 \\
&= (-1)^{n+1-0}.0 + \sum_{k=1}^{n+1}(-1)^{n+1-k}k^2 \\
&= \sum_{k+1=1}^{n+1}(-1)^{n+1-(k+1)}(k+1)^2 \\
&= \sum_{k=0}^{n}(-1)^{n-k}(k^2+2k+1) \\
&= \sum_{k=0}^{n}(-1)^{n-k}k^2 + \sum_{k=0}^{n}(-1)^{n-k}2k + \sum_{k=0}^{n}(-1)^{n-k} \\
&= U_n + 2T_n + S_n
\end{aligned}$$

Split off the last term

$$
\begin{aligned}
U_{n+1} &= \sum_{k=0}^{n+1} (-1)^{n+1-k} k^2 \\
&= \sum_{k=0}^{n} (-1)^{n+1-k} k^2 + (-1)^{n+1-(n+1)} (n+1)^2 \\
&= -\sum_{k=0}^{n} (-1)^{n-k} k^2 + (n+1)^2 \\
&= -U_n + (n+1)^2 \\
&= (n+1)^2 - U_n
\end{aligned}
$$

From the above two equations

$$
\begin{aligned}
U_n + 2T_n + S_n &= (n+1)^2 - U_n \\
\implies U_n &= \frac{1}{2}((n+1)^2 - 2T_n - S_n) \\
\implies U_n &= \frac{1}{2}(n^2 + n)
\end{aligned}
$$

15 Problem 23

Evaluate the sum $\sum_{k=1}^{n} \frac{2k+1}{k(k+1)}$ in two ways

a. Replace $\frac{1}{k(k+1)}$ by partial fraction

b. Sum by parts

15.1 Solution 23

$\sum_{k=1}^{n} \frac{2k+1}{k(k+1)}$

$\frac{2k+1}{k(k+1)} = \frac{A}{k} + \frac{B}{k+1}$

$\frac{2k+1}{k(k+1)} = \frac{A(k+1)+Bk}{k(k+1)}$

$\frac{2k+1}{k(k+1)} = \frac{(A+B)k+A}{k(k+1)}$

Comparing both the sides A = 1; A + B = 2;

A=1, B= 1

$\frac{2k+1}{k(k+1)} = \frac{1}{k} + \frac{1}{k+1}$

$\sum_{k=1}^{n} [\frac{1}{k} + \frac{1}{k+1}] = \sum_{k=1}^{n} \frac{1}{k} + \sum_{k=1}^{n} \frac{1}{k+1}$

$[1 + \frac{1}{2} + \frac{1}{3} + ... + \frac{1}{n}] + [\frac{1}{2} + \frac{1}{3} + ... + \frac{1}{n} + \frac{1}{n+1}]$

$H_n + H_{n-1} + \frac{1}{n+1}$

$2H_n - \frac{n}{n+1}$

(b)

$\sum_{k=1}^{n} \frac{2k+1}{k(k+1)} = \sum_{k=1}^{n+1} \frac{2k+1}{k(k+1)} dk$

$\sum u \triangle v = uv - \sum Ev \triangle u$

Let u(k) = 2k + 1;

$\triangle\ u(k) = 2;$

$\triangle\ v(k) = \frac{1}{k(k+1)} = (k-1)^{\underline{-2}}$

v(k) = $-(k-1)^{\underline{-1}} = -\frac{1}{k}$

$Ev = \frac{-1}{k+1}$

$\sum \frac{2k+1}{k(k+1)} dk = (2k+1)(\frac{-1}{k}) - \sum(\frac{-1}{k+1})2dk$

$2\sum(k^{\underline{-1}}dk - \frac{2k+1}{k})$

$2H_k - 2 - \frac{1}{k} + c$

$[\sum x^{\underline{m}} dx = H_x, \text{ if } m = -1]$

$\sum_{k=1}^{n+1} \frac{2k+1}{k(k+1)} dk = 2H_k - 2 - \frac{1}{k} + c\ |_1^{n+1}$

$[2H_{n+1} - 2 - \frac{1}{n+1} + c] - [2H_1 - 2 - 1 + c]$

$2H_n + \frac{2}{n+1} - 2 - \frac{1}{n+1} - 2 + 2 + 1$

$2H_n + \frac{1}{n+1} - 1$

$2H_n - \frac{n}{n+1}$

16 Problem 27

Compute $\triangle(c^{\underline{x}})$ and use it to deduce the value of $\sum_{k=1}^{n} \frac{(-2)^{\underline{k}}}{k}$.

16.1 Solution 27

We know that, $\boxed{\triangle f(x) = f(x+1) - f(x)}$

Also, we know, $\boxed{x^{\underline{m}} = \underbrace{x(x-1)(x-2)\cdots(x-m+1)}_{\text{m factors}}}$

Thus using the above formulae, we can derive the value of $\triangle(c^{\underline{x}})$ as,

$\triangle(c^{\underline{x}}) = c^{\underline{x+1}} - c^{\underline{x}}$

Now,
$c^{\underline{x+1}} = \underbrace{c(c-1)(c-2)\cdots(c-x)}_{\text{x+1 factors}}$
$c^{\underline{x}} = \underbrace{c(c-1)(c-2)\cdots(c-x+1)}_{\text{x factors}}$
Substituting the values of $c^{\underline{x+1}}$ and $c^{\underline{x}}$ in the equation for $\triangle(c^{\underline{x}})$, we get,

$$\begin{aligned}
\triangle(c^{\underline{x}}) &= (\mathbf{c(c-1)(c-2)\cdots(c-x+1)}(c-x)) - (\mathbf{c(c-1)(c-2)\cdots(c-x+1)}) \\
&= (\mathbf{c(c-1)(c-2)\cdots(c-x+1)}(c-x)) - (\mathbf{c(c-1)(c-2)\cdots(c-x+1)}) \\
&= (c(c-1)(c-2)\cdots(c-x+1))(c-x-1) \\
&= \frac{c(c-1)(c-2)\cdots(c-x+1)(\mathbf{c-x})(c-x-1)}{(\mathbf{c-x})} \\
&= \frac{c^{\underline{x+2}}}{(c-x)}
\end{aligned}$$

$\boxed{\text{Thus we have derived the relation, } \triangle(c^{\underline{x}}) = \frac{c^{\underline{x+2}}}{(c-x)}}$

In order to deduce the value of $\sum_{k=1}^{n} \frac{(-2)^{\underline{k}}}{k}$ using the calculated value of $\triangle(c^{\underline{x}})$, we substitute c = -2 and x = x-2, in the above equation, we get,

$$\begin{aligned}
\triangle((-2)^{\underline{x-2}}) &= \frac{(-2)^{\underline{(x-2)+2}}}{(-2-(x-2))} \\
&= \frac{(-2)^{\underline{x}}}{-x}
\end{aligned}$$

Before proceeding further we will prove an interesting fact, $-\triangle(f(x)) = \triangle(-f(x))$.

$$\begin{aligned}
-\triangle(f(x)) &= -(f(x+1) - f(x)) \\
&= -f(x+1) + f(X) \\
&= -f(x+1) - (-f(x)) \\
&= \triangle(-f(x))
\end{aligned}$$

Hence, we can say, $\triangle((-2)^{\underline{x-2}}) = \frac{(-2)^{\underline{x}}}{x}$.

Now, $\sum_a^b g(x)\delta(x) = \sum_{k=a}^{b-1} g(k)$ for all integers $b \geq a$.

Since, $\sum_{k=1}^{n} \frac{(-2)^{\underline{k}}}{k}$ is of the form $\sum_{k=a}^{b-1} g(k)$, we get,

$\sum_{k=1}^{n} \frac{(-2)^{\underline{k}}}{k} = \sum_{k=1}^{n+1} \frac{(-2)^{\underline{k}}}{k} \delta k$ for $n \geq 0$.

We know that $g(x) = \Delta(f(x))$ iff $\Sigma g(x)\delta x = f(x) + c$.
Putting the values derived in the above equations,

$$\begin{aligned}
\sum_{k=1}^{n+1} \frac{(-2)^{\underline{k}}}{k} \delta k &= -(-2)^{\underline{k-2}}\Big|_1^{n+1} \\
&= -(-2)^{\underline{n+1-2}} - [-(-2)^{\underline{1-2}}] \\
&= -(-2)^{\underline{n-1}} - [-(-2)^{\underline{-1}}] \\
&= -(-2)^{\underline{-1}} - (-2)^{\underline{n-1}} \\
&= \frac{1}{-2+1} - (-2)^{\underline{n-1}} \\
&= -1 - ((-2)(-2-1)(-2-2)\cdots(-2-(n-2))) \\
&= -1 - ((-2)(-3)(-4)\cdots(-n)) \\
&= -1 + ((-1)(-2)(-3)\cdots(-n)) \\
&= -1 + (-1)^n n!
\end{aligned}$$

We can verify the result for several values for n, and check with the form in the question and what our formula derives. For example, for n = 1,2,3 and 4 we get -2, 1, -7 and 23 respectively.

17 Problem 29

Evaluate the sum

$$\sum_{k=1}^{n} \frac{(-1)^k k}{4k^2-1}$$

17.1 Solution 29

We have

$$\begin{aligned}
S &= \sum_{k=1}^{n} \frac{(-1)^k k}{4k^2-1} \\
&= \sum_{k=1}^{n} \frac{(-1)^k k}{(2k-1)(2k+1)} \\
&= \sum_{k=1}^{n} (-1)^k \left(\frac{A}{2k-1} + \frac{B}{2k+1} \right) \qquad \text{Partial Fractions}
\end{aligned}$$

We find A and B.

$$\begin{aligned}
&\frac{k}{(2k+1)(2k-1)} = \frac{A}{(2k-1)} + \frac{B}{(2k+1)} \\
&\implies k = (2k+1)A + (2k-1)B \\
&\implies 2A + 2B = 1 \text{ and } A - B = 0 \\
&\implies A = B = \frac{1}{4}
\end{aligned}$$

Therefore

$$
\begin{aligned}
S &= \sum_{k=1}^{n} (-1)^k \left(\frac{A}{2k-1} + \frac{B}{2k+1} \right) \\
&= \sum_{k=1}^{n} (-1)^k \left(\frac{1/4}{2k-1} + \frac{1/4}{2k+1} \right) \\
&= \frac{1}{4} \sum_{k=1}^{n} (-1)^k \left(\frac{1}{2k-1} + \frac{1}{2k+1} \right) \\
&= \frac{1}{4} \left(\sum_{k=1}^{n} \frac{(-1)^k}{2k-1} + \sum_{k=1}^{n} \frac{(-1)^k}{2k+1} \right) \\
&= \frac{1}{4} \left(\frac{(-1)^1}{2.1-1} + \sum_{k=2}^{n} \frac{(-1)^k}{2k-1} + \sum_{k=1}^{n-1} \frac{(-1)^k}{2k+1} + \frac{(-1)^n}{2n+1} \right) \\
&= \frac{1}{4} \left(-1 + \sum_{k=2}^{n} \frac{(-1)^k}{2k-1} + \sum_{k=1}^{n-1} \frac{(-1)^k}{2k+1} + \frac{(-1)^n}{2n+1} \right) \\
&= \frac{1}{4} \left(-1 + \sum_{k+1=2}^{n} \frac{(-1)^{k+1}}{2(k+1)-1} + \sum_{k=1}^{n-1} \frac{(-1)^k}{2k+1} + \frac{(-1)^n}{2n+1} \right) \\
&= \frac{1}{4} \left(-1 - \sum_{k+1=2}^{n} \frac{(-1)^k}{2k+1} + \sum_{k=1}^{n-1} \frac{(-1)^k}{2k+1} + \frac{(-1)^n}{2n+1} \right) \\
&= \frac{1}{4} \left(-1 - \sum_{k=1}^{n-1} \frac{(-1)^k}{2k+1} + \sum_{k=1}^{n-1} \frac{(-1)^k}{2k+1} + \frac{(-1)^n}{2n+1} \right) \\
&= \frac{1}{4} \left(-1 + \frac{(-1)^n}{2n+1} \right) \\
&= \frac{-1}{4} + \frac{(-1)^n}{8n+4}
\end{aligned}
$$

18 Problem 31

Riemann zeta function $R(k)$ is defined to be the infinite sum

$$1 + \frac{1}{2^k} + \frac{1}{3^k} + \cdots = \sum_{j \geq 1} \frac{1}{j^k}$$

a) Prove that $\sum_{k \geq 2}(R(k) - 1) = 1$
b) What is the value of $\sum_{k \geq 1}(R(2k) - 1)$?

18.1 Solution 31

a) We have

$$\begin{aligned} S &= \sum_{k \geq 2}(R(k) - 1) \\ &= (R(2) - 1) + (R(3) - 1) + (R(4) - 1) + \cdots \end{aligned}$$

From the formula,

$$\begin{aligned} R(k) - 1 &= \sum_{j \geq 1} \frac{1}{j^k} \\ &= \frac{1}{2^k} + \frac{1}{3^k} + \frac{1}{4^k} + \cdots \end{aligned}$$

Therefore,

$$\begin{aligned} R(2) - 1 &= \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots \\ R(3) - 1 &= \frac{1}{2^3} + \frac{1}{3^3} + \frac{1}{4^3} + \cdots \\ R(3) - 1 &= \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} + \cdots \end{aligned}$$

so on

If we add along the columns we get

$$
\begin{aligned}
S &= (R(2)-1)+(R(3)-1)+(R(4)-1)+\cdots \\
&= \left(\frac{1}{2^2}+\frac{1}{2^3}+\frac{1}{2^4}+\cdots\right)+\left(\frac{1}{3^2}+\frac{1}{3^3}+\frac{1}{3^4}+\cdots\right) \\
&+\left(\frac{1}{4^2}+\frac{1}{4^3}+\frac{1}{4^4}+\cdots\right)+\cdots \\
&= \sum_{k=1}^{\infty}\frac{1}{2^k}+\sum_{k=1}^{\infty}\frac{1}{3^k}+\sum_{k=1}^{\infty}\frac{1}{4^k}+\cdots \\
&= \sum_{n=2}^{\infty}\left(\sum_{k=1}^{\infty}\frac{1}{n^k}\right) \\
&= \sum_{n=2}^{\infty}\left(\frac{\frac{1}{n^2}}{1-\frac{1}{n}}\right) \qquad \left(a+ar+ar^2+\cdots=\frac{a}{1-r},\ \text{when } r<1\right) \\
&= \sum_{n=2}^{\infty}\frac{1}{n(n-1)} \\
&= \sum_{n=2}^{\infty}\left(\frac{1}{n-1}-\frac{1}{n}\right) \\
&= \left(\frac{1}{1}-\frac{1}{2}\right)+\left(\frac{1}{2}-\frac{1}{3}\right)+\left(\frac{1}{3}-\frac{1}{4}\right)+\cdots \\
&= 1
\end{aligned}
$$

b) Let

$$
\begin{aligned}
S &= \sum_{k\geq 1}(R(2k)-1) \\
&= (R(2)-1)+(R(4)-1)+(R(6)-1)+\cdots
\end{aligned}
$$

From the formula,

$$
\begin{aligned}
R(k)-1 &= \sum_{j\geq 1}\frac{1}{j^k} \\
&= \frac{1}{2^k}+\frac{1}{3^k}+\frac{1}{4^k}+\cdots
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
R(2)-1 &= \frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\cdots \\
R(4)-1 &= \frac{1}{2^4}+\frac{1}{3^4}+\frac{1}{4^4}+\cdots \\
R(6)-1 &= \frac{1}{2^6}+\frac{1}{3^6}+\frac{1}{4^6}+\cdots \\
&\text{so on}
\end{aligned}
$$

If we add along the columns we get

$$\begin{aligned}
S &= (R(2)-1)+(R(4)-1)+(R(6)-1)+\cdots \\
&= \left(\frac{1}{2^2}+\frac{1}{2^4}+\frac{1}{2^6}+\cdots\right)+\left(\frac{1}{3^2}+\frac{1}{3^4}+\frac{1}{3^6}+\cdots\right) \\
&+\left(\frac{1}{4^2}+\frac{1}{4^4}+\frac{1}{4^6}+\cdots\right)+\cdots \\
&= \sum_{k=1}^{\infty}\frac{1}{2^{2k}}+\sum_{k=1}^{\infty}\frac{1}{3^{2k}}+\sum_{k=1}^{\infty}\frac{1}{4^{2k}}+\cdots \\
&= \sum_{n=2}^{\infty}\left(\sum_{k=1}^{\infty}\frac{1}{n^{2k}}\right) \\
&= \sum_{n=2}^{\infty}\left(\frac{\frac{1}{n^2}}{1-\frac{1}{n^2}}\right) \qquad \left(a+ar+ar^2+\cdots=\frac{a}{1-r},\text{ when } r<1\right) \\
&= \sum_{n=2}^{\infty}\frac{1}{(n^2-1)} \\
&= \sum_{n=2}^{\infty}\frac{1}{(n-1)(n+1)} \\
&= \frac{1}{2}\sum_{n=2}^{\infty}\left(\frac{1}{n-1}-\frac{1}{n+1}\right) \\
&= \frac{1}{2}\left(\left(\frac{1}{1}-\frac{1}{3}\right)+\left(\frac{1}{2}-\frac{1}{4}\right)+\left(\frac{1}{3}-\frac{1}{5}\right)+\cdots\right) \\
&= \frac{1}{2}\left(1+\frac{1}{2}\right) \\
&= \frac{3}{4}
\end{aligned}$$