Cse547

Chapter 1
problem 6

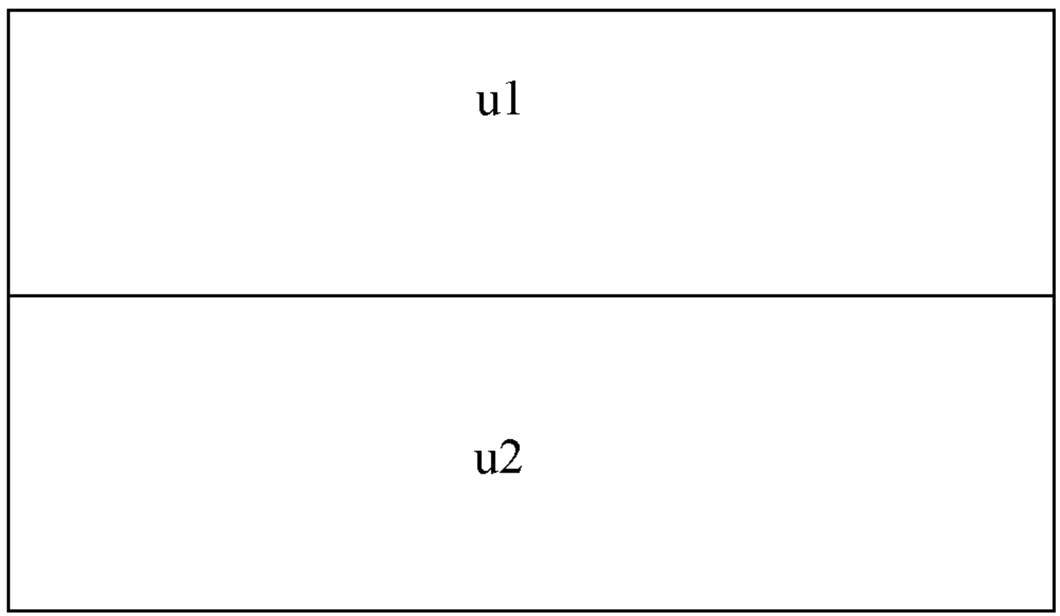
Question 6

Some of the regions defined by n lines in the plane are infinite, while others are bounded. What’s the maximum possible number of bounded regions?

First, as in most recursion questions, let’s try to analyze the questions for small values of n and then see if we can make generalizations about the question.

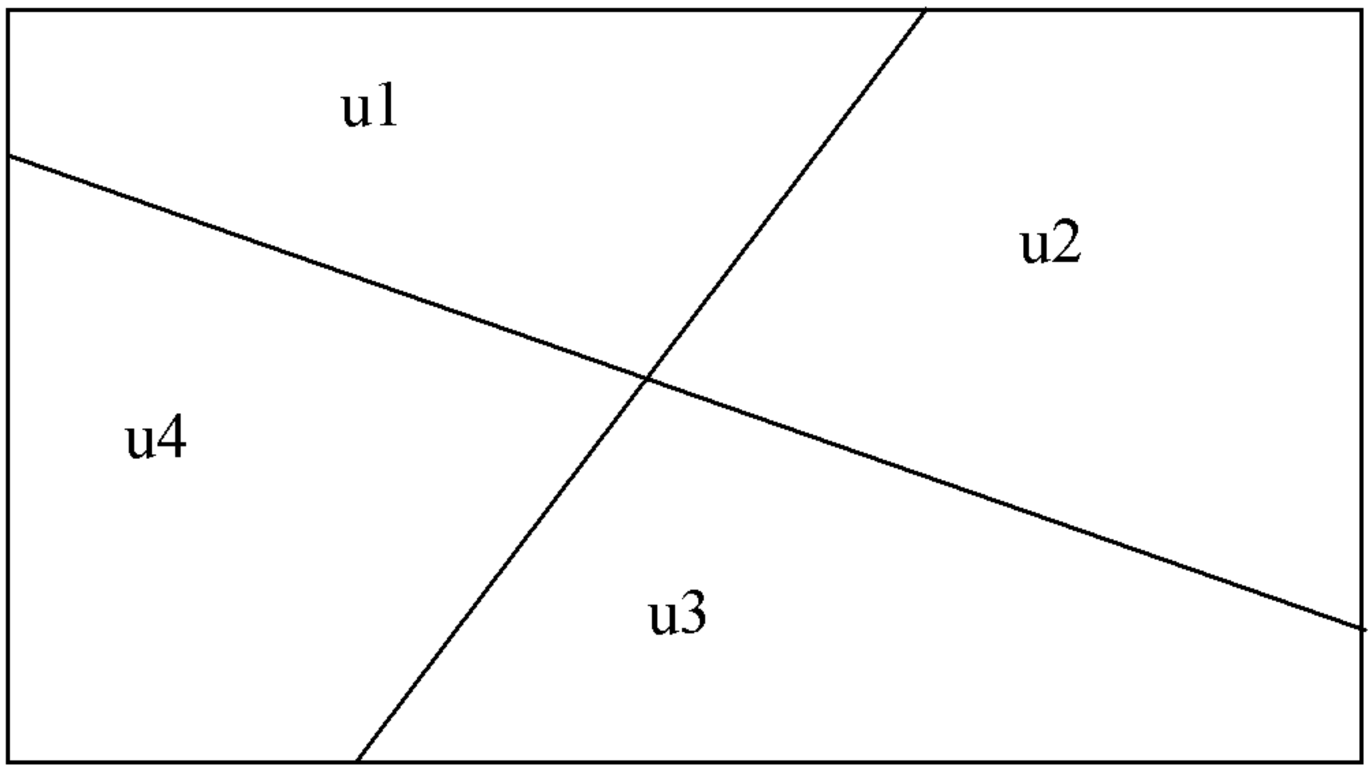
$n = 1$

As you can see there are 2 unbounded regions and no bounded regions.



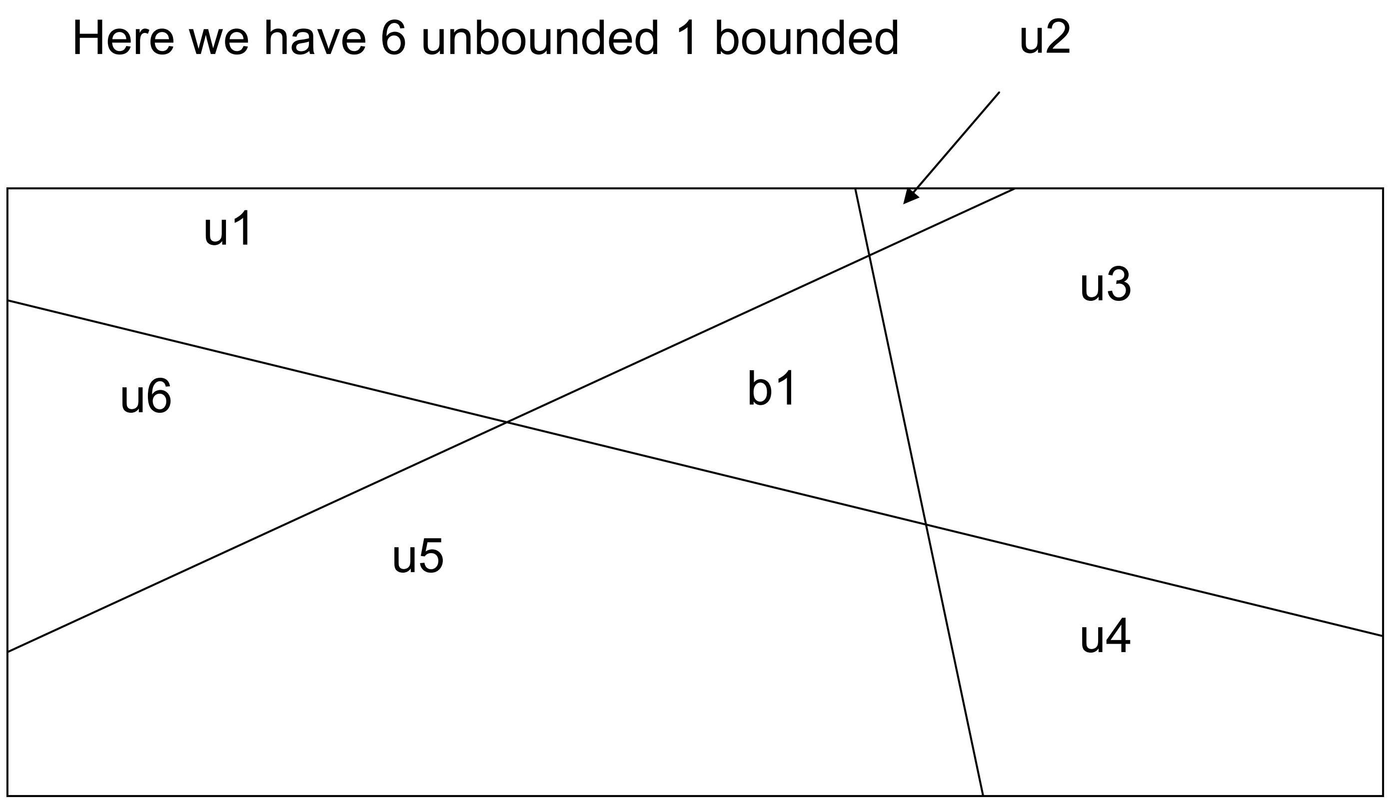
$n = 2$

Here we have 4 unbounded regions and still zero bounded regions.



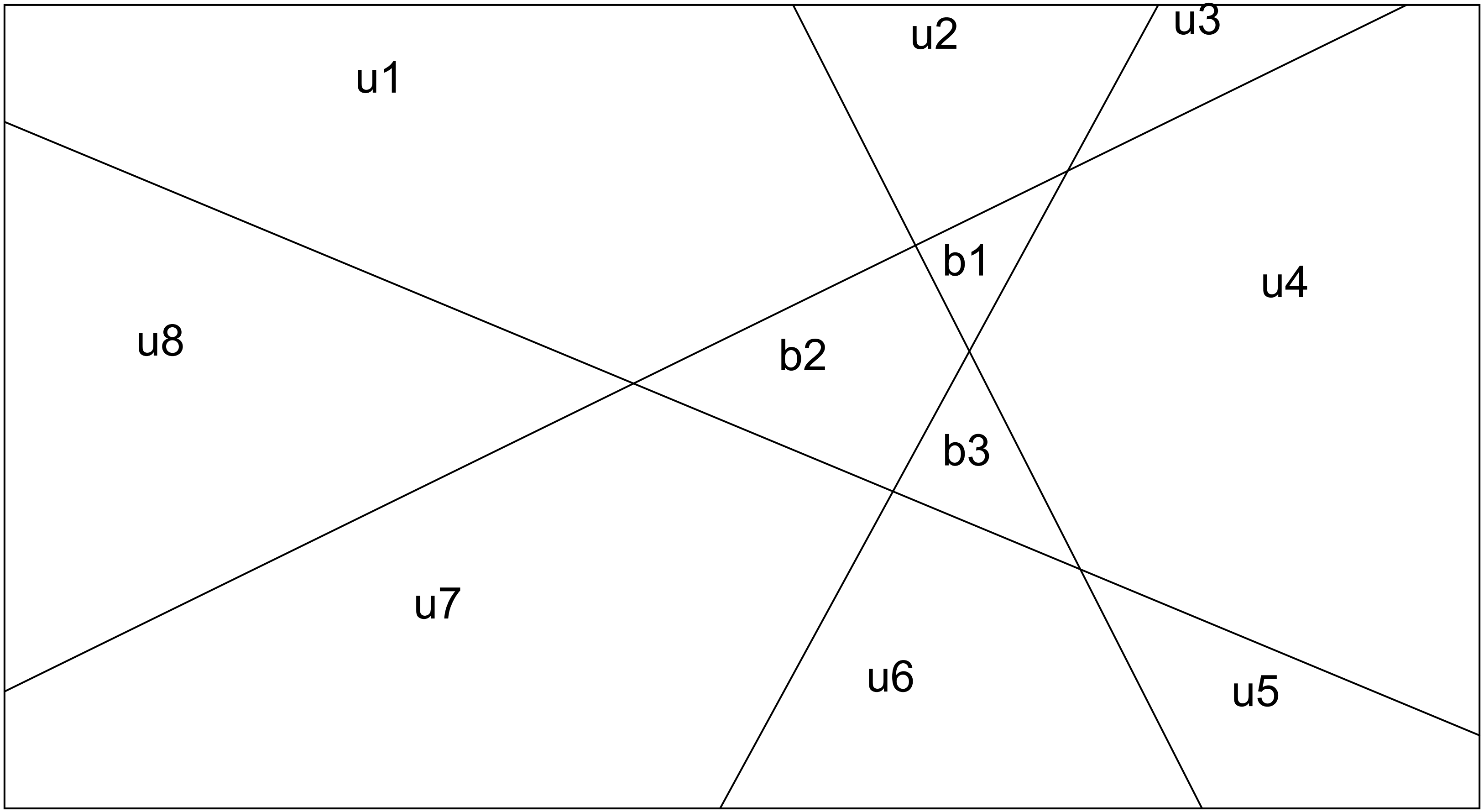
n = 3

Here we have 6 unbounded 1 bounded



n = 4

Now we have 8 unbounded and 3 bounded regions



Let’s now compare n, Ln (which is the maximum number of regions defined by n lines) and Bn, which we will call the maximum number of bounded regions defined by n lines.

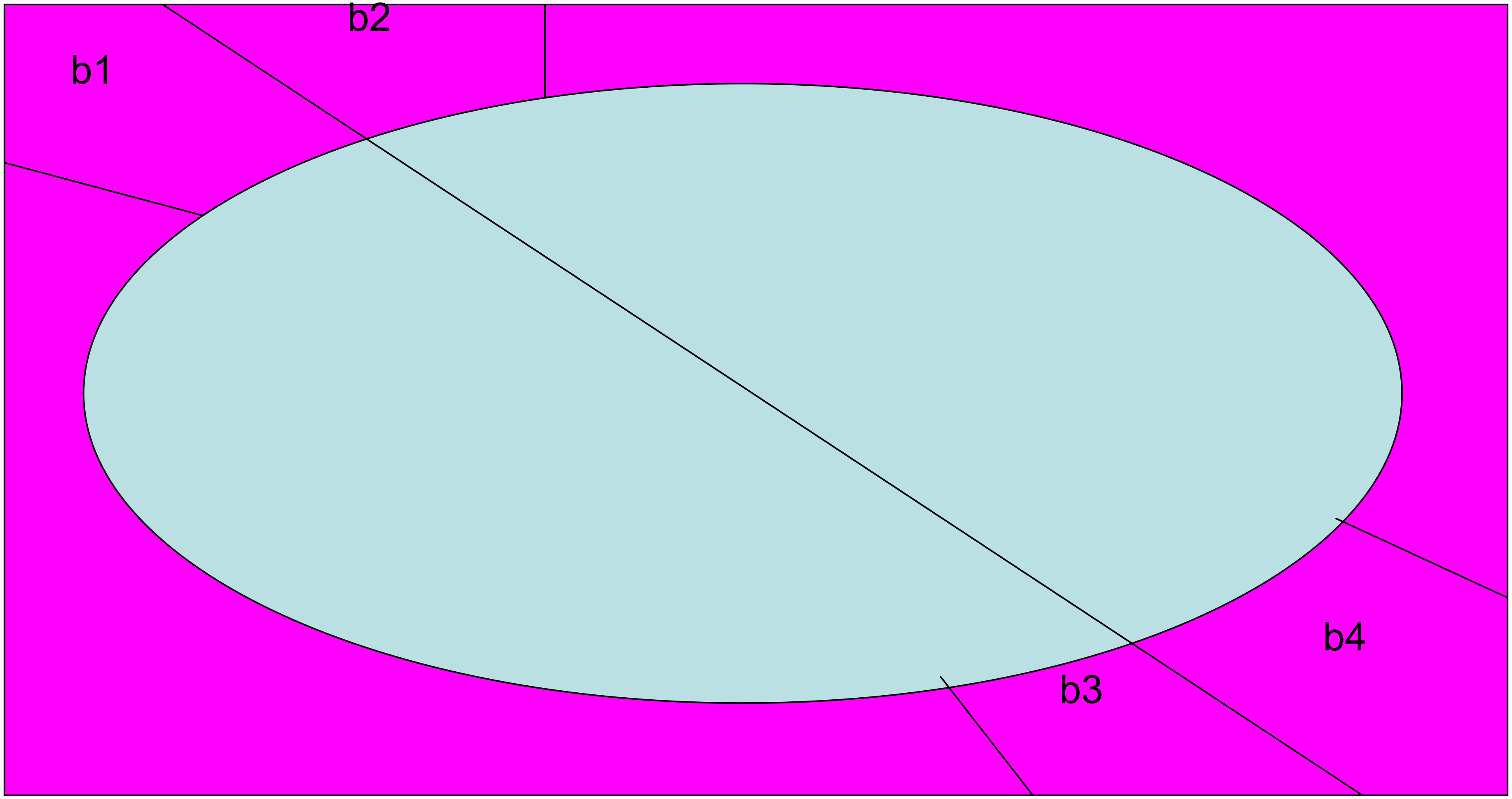
n	1	2	3	4	5
Ln	2	4	7	11	16
Bn	0	0	1	3	6

First thing to note, is that we are trying to find the MAXIMUM number of bounded regions possible. The maximum number will be defined by n non-parallel lines. As it states in the book, we know that the maximum number of regions, Ln, that can be formed is given by the recursion:

$$L_n = L_{n-1} + n$$

Now we just have to figure out, out of the n new regions, how many can possibly be bounded?

Below is a depiction of what happens when we add the n^{th} line. The circle represents the region where all of the previous intersection points lie.



As you can see from the picture on the previous slide, the n^{th} line can at a maximum cut 2 unbounded regions. That means that of the n new regions introduced, n – 2 of them must be new bounded regions. That hints at the following recursion:

$$B_0 = 0$$

$$B_1 = 0$$

$$B_2 = 0$$

$$B_n = B_{n-1} + (n - 2) \text{ , } n > 2$$

Closed Formula

$B_n = B_{n-1} + (n – 2)$

$B_n = B_{n-2} + (n – 3) + (n – 2)$

$B_n = B_{n-3} + (n – 4) + (n – 3) + (n – 2)$

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$$B_n = B_0 + B_1 + B_2 + \sum_{k=1}^{n} k$$

$B_n = S_{n-2}$

In general,

$$S_n = \frac{(n)(n+1)}{2}$$

So, since $B_n = S_{n-2}$:

$$B_n = \frac{(n-2)(n-1)}{2}$$

This is our predicted closed form solution. Proof by induction will verify it.

PROOF BY INDUCTION

BASE CASE:

$B_3 = 1$

$$B_3 = \frac{(3 - 2)(3 - 1)}{2} = 1 * 2 / 2 = 1$$

INDUCTIVE HYPOTHESIS

Assume that for all $n < k$:

$$B_n = \frac{(n - 2)(n - 1)}{2}$$

In particular;

$$B_{k-1} = \frac{(k - 3)(k - 2)}{2}$$

INDUCTIVE PROOF

Now we need to show, through the inductive hypothesis and the recursion that:

$$B_k = \frac{(k-2)(k-1)}{2}$$

$B_k = B_{k-1} + (k-2)$

$$B_k = \frac{(k-3)(k-2)}{2} + \frac{2(k-2)}{2}$$

$$B_k = \frac{k^2 - 3k + 2}{2}$$

$$B_k = \frac{(k-2)\,(k-1)}{2}$$

Another Method

Another way of looking at this problem is to notice that every line creates 2 unbounded regions. That would mean that for all n lines, the number of unbounded regions = 2n.

Since:

Bounded Regions = Regions – Unbounded Regions

$$B_n = L_n - 2n$$

$$B_n = \frac{n(n + 1)}{2} - 2n$$

$$B_n = \frac{(n - 2)(n - 1)}{2}$$