CHAPTER 1

Problem 20

SOLUTION

Problem

Use the repertoire method to solve the general five-parameter recurrence RF

Solve means here FIND the closed formula CF equivalent to RF

$$
\begin{aligned}
h(1) &= \alpha; \\
h(2n+0) &= 4h(n) + \gamma_0 n + \beta_0; \\
h(2n+1) &= 4h(n) + \gamma_1 n + \beta_1, \quad \textit{for all} \quad n \geq 1.
\end{aligned}
$$

General Form of CF

Our RF for h is a FIVE parameters function and it is a generalization of the General Josephus GJ f considered before

So we guess that now the general form of the CF is also a generalization of the one we already proved for GJ , i.e.

General form of CF is

$$h(n) = \alpha A(n) + \gamma_0 B(n) + \gamma_1 C(n) + \beta_0 D(n) + \beta_1 E(n)$$

The Problem 20 asks us to use repertoire method to prove that CF equivalent to RF.

Thinking Time

Solution requires a system of 5 equations on A(n), B(n), C(n), D(n), E(n) and accordingly a 5 repertoire function!

Let's THINK a bit before we embark on quite complicated calculations and without certainty that they would succeed (look at the Problem 16!)

First : we observe that when when $\gamma_0 = \gamma_1 = 0$, we get that h becomes for Generalize Josephus function f below for k = 4

$$f(1) = \alpha, \quad f(2n + j) = kf(n) + \beta_j,$$

where $k \geq 2$, $j = 0, 1$ and $n \geq 0$

It seems worth to examine the case $\gamma_0 = \gamma_1 = 0$ first.

GJ f Closed Formula Solution

We proved that GJ f has a relaxed k- representation closed formula

$$f((1, b_{m-1}, ...b_1, b_0)_2) = (\alpha, \beta_{b_{m-1}}, ...\beta_{b_0})_k$$

where β_{b_j} are defined by

$$\beta_{b_j} = \begin{cases} \beta_0 & b_j = 0 \\ \beta_1 & b_j = 1 \end{cases} \quad ; \quad j = 0, ..., m-1,$$

for relaxed k- radix representation defined as

$$(\alpha, \beta_{b_{m-1}}, ..., \beta_{b_0})_k = \alpha k^m + k^{m-1}\beta_{m-1} + ... + \beta_{b_0}$$

Special Case of h

Consider now a special case of our h, when $\gamma_0 = \gamma_1 = 0$

We know that it now has a relaxed 4 - representation closed formula

$$h((1, b_{m-1}, ...b_1, b_0)_2) = (\alpha, \beta_{b_{m-1}}, ...\beta_{b_0})_4$$

It means that we get

Fact 0 For any $n = (1, b_{m-1}, ...b_1, b_0)_2$,

$$h(n) = (\alpha, \beta_{b_{m-1}}, ...\beta_{b_0})_4$$

Observe that our general form of CF in this case becomes

$$h(n) = \alpha A(n) + \beta_0 D(n) + \beta_1 E(n)$$

We must have h(n) = h(n), for all n, so from this and Fact 0 we get the following equation 1.

Equation 1

Fact 1 For any $n = (1, b_{m-1}, ...b_1, b_0)_2$,

$$\alpha A(n) + \beta_0 D(n) + \beta_1 E(n) = (\alpha, \beta_{b_{m-1}}, ...\beta_{b_0})_4$$

This provides us with the Equation 1 for finding our general form of CF

Next Observation

Observe that A(n) in the Original Josephus was given (and proved to be) by a formula

$A(n) = 2^k$, for all $n = 2^k + \ell, 0 \leq \ell < 2^k$,

So we wonder if we could have a similar solution for our A(n)

Special Case of h

We evaluate now few initial values for h in case $\gamma_0 = \gamma_1 = 0$

$$
\begin{aligned}
h(1) &= \alpha; \\
h(2) &= h(2(1)+0) = 4h(1) + \beta_0 \\
&= 4\alpha + \beta_0;
\end{aligned}
$$

$$
\begin{aligned}
h(3) &= h(2(1)+1) = 4h(1) + \beta_1 \\
&= 4\alpha + \beta_1;
\end{aligned}
$$

$$
\begin{aligned}
h(4) &= h(2(2)+0) = 4h(2) + \beta_0 \\
&= 16\alpha + 5\beta_0;
\end{aligned}
$$

Equation 2

It is pretty obvious that we do have a similar formula for A(n) as on the Original Josephus O.

We write it as our Fact 2 and get our Equation 2.

Fact 2 for all $n = 2^k + \ell, 0 \leq \ell < 2^k,\ n \in N - \{0\}$,

$$A(n) = 4^k$$

The proof is almost identical to the one in the OJ, we re-write is here for our case as an exercise.

Reminder

Reminder: we investigate the case when $\gamma_0 = \gamma_1 = 0$, i.e. now our formulas are

RF: $\quad h(1) = \alpha, h(2n + j) = 4h(n) + \beta_j,$

where $j = 0, 1$ and $n \geq 0$, and the closed formula is

CF: $\quad h(n) = \alpha A(n) + \beta_0 D(n) + \beta_1 E(n)$

Proof of the Equation 2

Consider now a further case $\beta_0 = \beta_1$ =0, and $\alpha = 1$, i.e.

$RF: \quad h(1) = 1, \qquad h(2n) = 4h(n), \quad h(2n+1) = 4h(n)$

and $CF: \quad h(n) = A(n)$

We use $h(n) = A(n)$ and re-write RF in terms of A(n)

$AR: \quad A(1) = 1, \qquad A(2n) = 4A(n), \quad A(2n+1) = 4A(n)$

FACT: Closed formula CAR for AR is:

CAR: $A(n) = A(2^k + \ell) = 4^k, \quad 0 \leq l < 2^k$

Observe that this FACT is equivalent to our Fact 2, i.e. to validity of the Equation 2, so we are now proving

Fact 2 for all $n = 2^k + \ell, 0 \leq \ell < 2^k$,

$$A(n) = 4^k$$

Proof of the Equation 2

Proof by induction on k

BASE: k=0 i.e n=$2^0 + l, 0 \leq l < 1$, n = 1 and AR: A(1) = 1, CAR: A(1) = 4^0 = 1, and AR = CAR

Inductive Assumption:

$A(2^{k-1} + l)$ = $A(2^{k-1} + l) = 4^{k-1}, \quad 0 \leq l < 2^{k-1}$

Inductive Thesis:

$A(2^k + l)$ = $A(2^k + l) = 4^k, \; 0 \leq l < 2^k$

Two cases: $n \in even, \quad n \in odd$

C1: $n \in even$

n := 2n, and we have $2^k + l = 2n$ iff $l \in even$

Proof of the Equation 2

We evaluate n:

$2n = 2^k + l, \quad n = 2^{k-1} + \frac{l}{2}$

We use n in the inductive step. Observe that the correctness of using $\frac{l}{2}$ follows from that fact that $l \in even$ so $\frac{l}{2} \in N$ and it can be proved formally like on the previous slides.

Proof:

$A(2n) =^{reprn} A(2^k + l) =^{evaln} 4A(2^{k-1} + \frac{l}{2}) =^{ind}$
$4 * 4^{k-1} = 4^k$

Proof of the Equation 2

C2: $n \in odd$

n:= 2n+1, and we have $2^k + l = 2n + 1$ iff $l \in odd$

We evaluate n:

$2n + 1 = 2^k + l, \quad n = 2^{k-1} + \frac{l-1}{2}$

We use n in the inductive step. Observe that the correctness of using $\frac{l-1}{2}$ follows from that fact that $l \in odd$ so $\frac{l-1}{2} \in N$

Proof:

$A(2n+1) =^{reprn} A(2^k + l) =^{evaln} 4A(2^{k-1} + \frac{l-1}{2}) =^{ind}$
$4 * 4^{k-1} = 4^k$

It ends the proof of the Fact 2: $A(n) = 4^k$

Repertoire Method

We return now to out original functions:

RF: $h(1) = \alpha, h(2n) = 4h(n) + \gamma_0 n + \beta_0,$
$h(2n+1) = 4h(n) + \gamma_1 n + \beta_1,$

CF: $h(n) = \alpha A(n) + \gamma_0 B(n) + \gamma_1 C(n) + \beta_0 D(n) + \beta_1 E(n)$

We have already developed 2 equations (Fact 1 and Fact 2) so we need now to consider only 3 repertoire functions

Repertoire Function 1

Consider a repertoire function 1: h(n) = 1, for all $n \in N - \{0\}$,

We have h(n) = 1, h(1)= α, so we get $\alpha = 1$ and we evaluate

$h(2n) = 4h(n) + \gamma_0 n + \beta_0$	$h(2n+1) = 4h(n) + \gamma_1 n + \beta_1$;
$1 = 4 + \gamma_0 n + \beta_0$	$1 = 4 + \gamma_1 n + \beta_1$
$0 = 3 + \gamma_0 n + \beta_0$	$0 = 3 + \gamma_1 n + \beta_1$

Solution: $\gamma_0 = \gamma_1 = 0, \quad \beta_0 = \beta_1 = -3$

Equation 3

CF: $h(n) = \alpha A(n) + \gamma_0 B(n) + \gamma_1 C(n) + \beta_0 D(n) + \beta_1 E(n)$

We evaluate CF for $\alpha = 1, \gamma_0 = \gamma_1 = 0, \quad \beta_0 = \beta_1 = -3$ and get

CF = RF iff the following holds

Fact 3 For all $n \in N - \{0\}$,

$$A(n) - 3D(n) - 3E(n) = 1$$

Repertoire Function 2

Consider a repertoire function 2: h(n) = n, for all $n \in N - \{0\}$

h(1) = α, h(1) = 1 and h(n)=h(n), hence $\alpha = 1$

$$h(2n) = 4h(n) + \gamma_0 n + \beta_0$$
$$2n = 4n + \gamma_0 n + \beta_0$$
$$0 = (\gamma_0 + 2)n + \beta_0$$

$$h(2n+1) = 4h(n) + \gamma_1 n + \beta_1;$$
$$2n + 1 = 4n + \gamma_1 n + \beta_1$$
$$0 = (\gamma_1 + 2)n + (\beta_1 - 1)$$

Solution: $\gamma_0 = \gamma_1 = -2$, $\beta_0 = 0$, $\beta_1 = 1$

Equation 4

CF: $h(n) = \alpha A(n) + \gamma_0 B(n) + \gamma_1 C(n) + \beta_0 D(n) + \beta_1 E(n)$

We evaluate CF for $\alpha = 1, \gamma_0 = \gamma_1 = -2,\ \ \beta_0 = 0,\ \beta_1 = 1$ and get

CF = RF iff the following holds

Fact 4 For all $n \in N - \{0\}$

$$A(n) - 2B(n) - 2C(n) + E(n) = n$$

Repertoire Function 3

Consider a repertoire function 3: $h(n) = n^2$, for all $n \in N$

h(1) = α, h(1) = 1, hence $\alpha = 1$

$h(2n+0) = 4h(n) + \gamma_0 n + \beta_0$	$h(2n+1) = 4h(n) + \gamma_1 n + \beta_1$;
$(2n)^2 = 4n^2 + \gamma_0 n + \beta_0$	$(2n+1)^2 = 4n^2 + \gamma_1 n + \beta_1$
$\cancel{4}n^2 = \cancel{4}n^2 + \gamma_0 n + \beta_0$	$4n^2 + 4n + 1 = 4n^2 + \gamma_1 n + \beta_1$
$0 = \gamma_0 n + \beta_0$	$0 = (\gamma_1 - 4)n + (\beta_1 - 1)$

Solution: $\gamma_0 = 0,\ \gamma_1 = 4,\ \beta_0 = 0,\ \beta_1 = 1$.

Equation 5

CF: $h(n) = \alpha A(n) + \gamma_0 B(n) + \gamma_1 C(n) + \beta_0 D(n) + \beta_1 E(n)$

We evaluate CF for $\alpha = 1,\ \gamma_0 = 0,\ \gamma_1 = 4,\ \beta_0 = 0,\ \beta_1 = 1$ and get

CF = RF iff the following holds

Fact 5 For all $n \in N - \{0\}$

$$A(n) + 4C(n) + E(n) = n^2$$

Repertoire Method System of Equations

We obtained the following system of 5 equations on A(n), B(n), C(n), D(n), E(n)

1. $\alpha A(n) + \beta_0 D(n) + \beta_1 E(n) = (\alpha, \beta_{b_{m-1}}, \ldots \beta_{b_0})_4$
2. $A(n) = 4^k$
3. A(n) - 3D(n) - 3E(n) = 1
4. A(n) - 2B(n) - 2 C(n) + E(n) = n
5. $A(n) + 4C(n) + E(n) = n^2$

We solve it and put the solution into

$h(n) = \alpha A(n) + \gamma_0 B(n) + \gamma_1 C(n) + \beta_0 D(n) + \beta_1 E(n)$