CHAPTER 1
Problem 20
SOLUTION
Problem

Use the repertoire method to solve the general five-parameter recurrence RF

Solve means here FIND the closed formula CF equivalent to RF

\[
\begin{align*}
h(1) & = \alpha; \\
h(2n + 0) & = 4h(n) + \gamma_0 n + \beta_0; \\
h(2n + 1) & = 4h(n) + \gamma_1 n + \beta_1, \quad \text{for all } n \geq 1.
\end{align*}
\]
General Form of CF

Our RF for $h$ is a FIVE parameters function and it is a generalization of the General Josephus GJ $f$ considered before.

So we guess that now the general form of the CF is also a generalization of the one we already proved for GJ, i.e.

General form of CF is

$$h(n) = \alpha A(n) + \gamma_0 B(n) + \gamma_1 C(n) + \beta_0 D(n) + \beta_1 E(n)$$

The Problem 20 asks us to use repertoire method to prove that CF equivalent to RF.
Thinking Time

Solution requires a system of 5 equations on $A(n)$, $B(n)$, $C(n)$, $D(n)$, $E(n)$ and accordingly a 5 repertoire function!

Let’s THINK a bit before we embark on quite complicated calculations and without certainty that they would succeed (look at the Problem 16!)

First: we observe that when when $\gamma_0 = \gamma_1 = 0$, we get that $h$ becomes for Generalize Josephus function $f$ below for $k = 4$

\[
f(1) = \alpha, \quad f(2n + j) = kf(n) + \beta_j,
\]

where $k \geq 2$, $j = 0, 1$ and $n \geq 0$

It seems worth to examine the case $\gamma_0 = \gamma_1 = 0$ first.
GJ f Closed Formula Solution

We proved that GJ f has a relaxed k- representation closed formula

\[ f((1, b_{m-1}, \ldots, b_1, b_0)_2) = (\alpha, \beta_{b_{m-1}}, \ldots \beta_{b_0})_k \]

where \( \beta_{b_j} \) are defined by

\[
\beta_{b_j} = \begin{cases} \\
\beta_0 & b_j = 0 \\
\beta_1 & b_j = 1
\end{cases} ; \quad j = 0, \ldots, m - 1,
\]

for relaxed k- radix representation defined as

\[
(\alpha, \beta_{b_{m-1}}, \ldots, \beta_{b_0})_k = \alpha k^m + k^{m-1} \beta_{m-1} + \ldots + \beta_{b_0}
\]
Special Case of $h$

Consider now a special case of our $h$, when $\gamma_0 = \gamma_1 = 0$

We know that it now has a relaxed 4 - representation closed formula

$$h((1, b_{m-1}, ... b_1, b_0)_2) = (\alpha, \beta b_{m-1}, ... \beta b_0)_4$$

It means that we get

**Fact 0**  For any $n = (1, b_{m-1}, ... b_1, b_0)_2$,

$$h(n) = (\alpha, \beta b_{m-1}, ... \beta b_0)_4$$

Observe that our general form of CF in this case becomes

$$h(n) = \alpha A(n) + \beta_0 D(n) + \beta_1 E(n)$$

We must have $h(n) = h(n)$, for all $n$, so from this and Fact 0 we get the following equation 1.
Fact 1  For any $n = (1, b_{m-1}, \ldots, b_1, b_0)_2$,

$$\alpha A(n) + \beta_0 D(n) + \beta_1 E(n) = (\alpha, \beta_{b_{m-1}}, \ldots, \beta_{b_0})_4$$

This provides us with the Equation 1 for finding our general form of CF.
Next Observation

Observe that $A(n)$ in the Original Josephus was given (and proved to be) by a formula

$$A(n) = 2^k, \text{ for all } n = 2^k + \ell, 0 \leq \ell < 2^k,$$

So we wonder if we could have a similar solution for our $A(n)$
Special Case of $h$

We evaluate now few initial values for $h$ in case $\gamma_0 = \gamma_1 = 0$

\begin{align*}
h(1) & = \alpha; \\
h(2) & = h(2(1) + 0) = 4h(1) + \beta_0 \\
& = 4\alpha + \beta_0; \\
h(3) & = h(2(1) + 1) = 4h(1) + \beta_1 \\
& = 4\alpha + \beta_1; \\
h(4) & = h(2(2) + 0) = 4h(2) + \beta_0 \\
& = 16\alpha + 5\beta_0;
\end{align*}
Equation 2

It is pretty obvious that we do have a similar formula for $A(n)$ as on the Original Josephus O. We write it as our Fact 2 and get our Equation 2.

**Fact 2** for all $n = 2^k + \ell, 0 \leq \ell < 2^k, n \in N - \{0\},$

$$A(n) = 4^k$$

The proof is almost identical to the one in the OJ, we re-write is here for our case as an exercise.
Reminder

Reminder: we investigate the case when $\gamma_0 = \gamma_1 = 0$, i.e. now our formulas are

RF: $h(1) = \alpha$, $h(2n + j) = 4h(n) + \beta_j$,

where $j = 0, 1$ and $n \geq 0$, and the closed formula is

CF: $h(n) = \alpha A(n) + \beta_0 D(n) + \beta_1 E(n)$
Proof of the Equation 2

Consider now a further case $\beta_0 = \beta_1 = 0$, and $\alpha = 1$, i.e.

$RF: \, h(1) = 1, \quad h(2n) = 4h(n), \quad h(2n + 1) = 4h(n)$
and $CF: \, h(n) = A(n)$

We use $h(n) = A(n)$ and re-write RF in terms of A(n)

$AR: \, A(1) = 1, \quad A(2n) = 4A(n), \quad A(2n + 1) = 4A(n)$

FACT: Closed formula CAR for AR is:

$CAR: \, A(n) = A(2^k + \ell) = 4^k, \quad 0 \leq \ell < 2^k$

Observe that this FACT is equivalent to our Fact 2, i.e. to validity of the Equation 2, so we are now proving

Fact 2 \ for all $n = 2^k + \ell, \ 0 \leq \ell < 2^k$,

\[ A(n) = 4^k \]
Proof of the Equation 2

**Proof** by induction on \( k \)

**BASE:** \( k=0 \) i.e \( n=2^0 + l, 0 \leq l < 1, \ n = 1 \) and **AR:** \( A(1) = 1 \), **CAR:** \( A(1) = 4^0 = 1 \), and **AR = CAR**

**Inductive Assumption:**
\[
A(2^{k-1} + l) = A(2^{k-1} + l) = 4^{k-1}, \quad 0 \leq l < 2^{k-1}
\]

**Inductive Thesis:**
\[
A(2^k + l) = A(2^k + l) = 4^k, \quad 0 \leq l < 2^k
\]

Two cases: \( n \in \text{even}, \ n \in \text{odd} \)

**C1:** \( n \in \text{even} \)

\( n := 2n \), and we have \( 2^k + l = 2n \) iff \( l \in \text{even} \)
Proof of the Equation 2

We evaluate $n$:

\[ 2n = 2^k + l, \quad n = 2^{k-1} + \frac{l}{2} \]

We use $n$ in the inductive step. Observe that the correctness of using $\frac{l}{2}$ follows from that fact that $l \in \text{even}$ so $\frac{l}{2} \in N$ and it can be proved formally like on the previous slides.

Proof:

\[
A(2n) =^\text{reprn} A(2^k + l) =^\text{evaln} 4A(2^{k-1} + \frac{l}{2}) =^\text{ind} 4 \times 4^{k-1} = 4^k
\]
Proof of the Equation 2

\textbf{C2: } n \in \text{odd} \\
n := 2n + 1, \text{ and we have } 2^k + l = 2n + 1 \text{ iff } l \in \text{odd} \\
We evaluate \( n \): \\
\( 2n + 1 = 2^k + l \), \quad n = 2^{k-1} + \frac{l-1}{2} \\
We use \( n \) in the inductive step. Observe that the correctness of using \( \frac{l-1}{2} \) follows from that fact that \( l \in \text{odd} \) so \( \frac{l-1}{2} \in N \\
Proof: \\
A(2n + 1) \overset{\text{reprn}}{=} A(2^k + l) \overset{\text{evaln}}{=} 4A(2^{k-1} + \frac{l-1}{2}) \overset{\text{ind}}{=} 4 \times 4^{k-1} = 4^k \\
It ends the proof of the Fact 2: A(n) = 4^k
Repertoire Method

We return now to our original functions:

RF: \[ h(1) = \alpha, \ h(2n) = 4h(n) + \gamma_0 n + \beta_0, \]
\[ h(2n + 1) = 4h(n) + \gamma_1 n + \beta_1, \]

CF: \[ h(n) = \alpha A(n) + \gamma_0 B(n) + \gamma_1 C(n) + \beta_0 D(n) + \beta_1 E(n) \]

We have already developed 2 equations (Fact 1 and Fact 2) so we need now to consider only 3 repertoire functions.
Consider a repertoire function 1: \( h(n) = 1 \), for all \( n \in \mathbb{N} - \{0\} \),

We have \( h(n) = 1 \), \( h(1) = \alpha \), so we get \( \alpha = 1 \) and we evaluate

\[
\begin{align*}
    h(2n) &= 4h(n) + \gamma_0 n + \beta_0 \\
    1 &= 4 + \gamma_0 n + \beta_0 \\
    0 &= 3 + \gamma_0 n + \beta_0 \\
    h(2n + 1) &= 4h(n) + \gamma_1 n + \beta_1 \\
    1 &= 4 + \gamma_1 n + \beta_1 \\
    0 &= 3 + \gamma_1 n + \beta_1 
\end{align*}
\]

Solution: \( \gamma_0 = \gamma_1 = 0 \), \( \beta_0 = \beta_1 = -3 \)
Equation 3

\[ h(n) = \alpha A(n) + \gamma_0 B(n) + \gamma_1 C(n) + \beta_0 D(n) + \beta_1 E(n) \]

We evaluate \( \text{CF} \) for \( \alpha = 1, \gamma_0 = \gamma_1 = 0, \beta_0 = \beta_1 = -3 \) and get

\( \text{CF} = \text{RF} \) iff the following holds

**Fact 3** For all \( n \in \mathbb{N} - \{0\}, \)

\[ A(n) - 3D(n) - 3E(n) = 1 \]
Consider a repertoire function 2: \( h(n) = n \), for all \( n \in \mathbb{N} - \{0\} \)
\( h(1) = \alpha \), \( h(1) = 1 \) and \( h(n)=h(n) \), hence \( \alpha = 1 \)

\[
\begin{align*}
    h(2n) &= 4h(n) + \gamma_0 n + \beta_0 \\
    2n &= 4n + \gamma_0 n + \beta_0 \\
    0 &= (\gamma_0 + 2)n + \beta_0 \\
    h(2n + 1) &= 4h(n) + \gamma_1 n + \beta_1; \\
    2n + 1 &= 4n + \gamma_1 n + \beta_1 \\
    0 &= (\gamma_1 + 2)n + (\beta_1 - 1)
\end{align*}
\]

Solution: \( \gamma_0 = \gamma_1 = -2 \), \( \beta_0 = 0 \), \( \beta_1 = 1 \)
Equation 4

\[ h(n) = \alpha A(n) + \gamma_0 B(n) + \gamma_1 C(n) + \beta_0 D(n) + \beta_1 E(n) \]

We evaluate \( \text{CF} \) for \( \alpha = 1, \gamma_0 = \gamma_1 = -2, \beta_0 = 0, \beta_1 = 1 \) and get

\( \text{CF} = \text{RF} \) iff the following holds

Fact 4  For all \( n \in N - \{0\} \)

\[ A(n) - 2B(n) - 2C(n) + E(n) = n \]
Consider a repertoire function 3: \( h(n) = n^2 \), for all \( n \in \mathbb{N} \)
\( h(1) = \alpha, \ h(1) = 1 \), hence \( \alpha = 1 \)

\[
\begin{align*}
\h(2n + 0) &= 4h(n) + \gamma_0 n + \beta_0 \\
(2n)^2 &= 4n^2 + \gamma_0 n + \beta_0 \\
\mathcal{A}n^2 &= \mathcal{A}n^2 + \gamma_0 n + \beta_0 \\
0 &= \gamma_0 n + \beta_0 \\
\h(2n + 1) &= 4h(n) + \gamma_1 n + \beta_1; \\
(2n + 1)^2 &= 4n^2 + \gamma_1 n + \beta_1 \\
4n^2 + 4n + 1 &= 4n^2 + \gamma_1 n + \beta_1 \\
0 &= (\gamma_1 - 4)n + (\beta_1 - 1)
\end{align*}
\]

Solution: \( \gamma_0 = 0, \ \gamma_1 = 4, \ \beta_0 = 0, \ \beta_1 = 1. \)
Equation 5

CF: \( h(n) = \alpha A(n) + \gamma_0 B(n) + \gamma_1 C(n) + \beta_0 D(n) + \beta_1 E(n) \)

We evaluate CF for \( \alpha = 1, \gamma_0 = 0, \gamma_1 = 4, \beta_0 = 0, \beta_1 = 1 \) and get

CF = RF \ iff \ the following holds

Fact 5 \ For all \( n \in N - \{0\} \)

\[ A(n) + 4C(n) + E(n) = n^2 \]
Repertoire Method System of Equations

We obtained the following system of 5 equations on $A(n)$, $B(n)$, $C(n)$, $D(n)$, $E(n)$

1. $\alpha A(n) + \beta_0 D(n) + \beta_1 E(n) = (\alpha, \beta_{b_{m-1}}, \ldots \beta_{b_0})_4$
2. $A(n) = 4^k$
3. $A(n) - 3D(n) - 3E(n) = 1$
4. $A(n) - 2B(n) - 2 C(n) + E(n) = n$
5. $A(n) + 4C(n) + E(n) = n^2$

We solve it and put the solution into

$h(n) = \alpha A(n) + \gamma_0 B(n) + \gamma_1 C(n) + \beta_0 D(n) + \beta_1 E(n)$