Chapter one

Problem 2 and 14
Problem 2

- Find the shortest sequence of moves that transfers a tower of \( n \) disks from the left peg A to right peg B, if direct moves between A and B are disallowed. Each move must be to or from the middle peg. As usual a larger disk must never appear above a smaller one.)
The problem can be solved recursively as follows:

First consider case, \( n=1 \), where we have to move a single disk from A to B. Since direct moves are disallowed this requires 2 moves and hence \( P(1) = 2 \).
We define the task of moving n disks from peg A to peg B recursively as follows.

- By assumption we know how to move the top $n-1$ disks from A to B without direct move $\rightarrow P(n-1)$
- Move the largest disk from A to the middle $\rightarrow 1$
- Again by assumption we know how to move the top $n-1$ disks from B to A $\rightarrow P(n-1)$
- Move the largest disk from the middle to B $\rightarrow 1$
- Again by assumption we know how to move the top $n-1$ disks from A to B without direct move $\rightarrow P(n-1)$
• After these moves all the n disks will be in order on peg B. Thus we can see that the total moves required to transfer the n disks is 
P(n) = 3P(n-1) + 2.
• We want to guess the close form so we look at small cases: where we know P(1)=2;
• P(2)=3*2 +2 =8,
• P(3)=3*8+2=26,..
• We suggest the solution to this recurrence as: P(n) =3^n -1.
Proof by induction for $P(n) = 3^n - 1$:

$P(0) = 3^0 - 1 = 0$

For $n > 0$ we assume that it works when $n$ is replaced by $n-1$:

$P(n) = 3P(n-1) + 2 = 3(3^{n-1} - 1) + 2 = 3^n - 1$. 
Problem 14
Problem 14

How many pieces of cheese can you obtain from a single thick piece by making five straight slices? (the cheese must stay in its original position while you do all the cutting, and each slice must correspond to a plane in 3D) Find a recurrence relation for \( p(n) \), the maximum number of three dimensional regions that can be defined by \( n \) different planes.
We use this problem

- How many slices of pizza can a person obtain by making \( n \) straight cuts with a pizza knife.

- Which actually is “What is the maximum number \( L(n) \) of regions defined by \( n \) lines in the plane?”.
  
  We showed by induction \( L(n) = L(n-1) + n \) for \( n > 0 \)

  And the close formula for that is \( L(n) = \frac{n(n+1)}{2} + 1 \)
• Consider the most general case, where planes inserted are not parallel and no set of more than 2 planes intersect in the same line. For the n’th plane, all the n-1 previously intersected planes will intersect the n’th plane and create n-1 cuts on that plane.

• As it was shown these n-1 cuts will divide the n’th plane into at most 1+ n(n-1)/2 regions.
The most of new pieces by the n’th cut is exactly this number.
Thus the recurrence relation that describe the maximum number of pieces attainable using n cuts is
\[ P(n) = P(n-1) + 1 + n(n-1)/2 \]
given that the base case \( P(0) = 1 \).
\[ P(1) = 2; \]
\[ P(2) = 2 + 1 + 1 = 4 = 1 \times 2 \times 3/6 + 2 + 1 \]
\[ P(3) = 4 + 3 + 1 = 8 = 2 \times 3 \times 4/6 + 3 + 1 \]
\[ P(4) = 8 + 6 + 1 = 15 = 3 \times 4 \times 5/6 + 4 + 1 \]
\[ P(5) = 15 + 10 + 1 = 26 = 4 \times 5 \times 6/6 + 5 + 1 \]
\[ P(6) = 26 + 15 + 1 = 42 = 5 \times 6 \times 7/6 + 6 + 1 \]
The solution to this recurrence is
\[ P(n) = (n-1)n(n+1)/6 + n + 1 \] which can be proved by induction.
Prove by induction: \( P(n)=(n-1)n(n+1)/6 + n+1 \)

It works for \( P(0)=1 \);
Assume it holds when \( n \) is replaced with \( n-1 \) since
\[
P(n) = P(n-1) + n(n-1)/2 + 1 \text{ then } P(n) = (n-2)(n-1)(n)/6 + n + n(n-1)/2 + 1
\]
And:
\[
P(n) = (n-1)n(n+1)/6 + n+1
\]