CSE547
Chapter 1, Problem 16

PROBLEM

Use the Repertoire method to solve the general four-parameter recurrence:

$$g(1) = \alpha$$

$$g(2n+j) = 3g(n) + \gamma n + \beta_j,$$

for $j = 0,1$ and $n \geq 1$ and $n \in N$

Hint : Try the function $g(n) = n$

What does “solve” mean?

By “Solving a recurrence” we mean that given a recursive function, our goal is to find a closed formula that is “equivalent” to the recursive function. In other words, the closed function must be defined on the same domain as the recursive function, and also both the recursive and closed functions must output the same values, when both of them are provided with the same input values.

SOLUTION

For the sake of clarity let us write the given recursive formula in the following way:

$$g(1) = \alpha$$
$$g(2n+0) = 3g(n) + \gamma n + \beta_0$$
$$g(2n+1) = 3g(n) + \gamma n + \beta_1$$
$$\text{(for } n \in \mathbb{N} \text{ and } n \geq 1)$$

Let “R” be the above recursive formula.

How should we proceed?

There are three main steps we need to take:

1) Find a few initial values for R.
2) Looking at the values obtained in step 1, find (guess) the general form for the closed formula.
3) Use Repertoire method to find the exact closed formula.

STEP 1: Finding Initial Values For “R”

Finding Few Initial Values For "R"

$g(1) = \alpha.$

$$g(2) = g(2(1) + 0) = 3g(1) + \gamma + \beta_0$$
$$= 3\alpha + \gamma + \beta_0 .$$

$$g(3) = g(2(1) + 1) = 3g(1) + \gamma + \beta_1$$
$$= 3\alpha + \gamma + \beta_1 .$$

$$g(4) = g(2(2) + 0) = 3g(2) + 2\gamma + \beta_0$$
$$= 9\alpha + 5\gamma + 4\beta_0 .$$

Evaluating “R” on some values (Continued)

$$g(5) = g(2(2) + 1) = 3g(2) + 2\gamma + \beta_1$$
$$= 9\alpha + 5\gamma + 3\beta_0 + \beta_1.$$

$$g(6) = g(2(3) + 0) = 3g(3) + 3\gamma + \beta_0$$
$$= 9\alpha + 6\gamma + \beta_0 + 3\beta_1.$$

$$g(7) = g(2(3) + 1) = 3g(3) + 3\gamma + \beta_1$$
$$= 9\alpha + 6\gamma + 4\beta_1.$$

$$g(8) = g(2(4) + 0) = 3g(4) + 4\gamma + \beta_0$$
$$= 27\alpha + 19\gamma + 13\beta_0 \,.$$

Evaluating "R" on some values (Continued)

$g(9) = g(2(4) + 1) = 3g(4) + 4\gamma + \beta_1$

$= 27\alpha + 19\gamma + 12\beta_0 + \beta_{1.}$

$g(10) = g(2(5) + 0) = 3g(5) + 5\gamma + \beta_0$

$= 27\alpha + 20\gamma + 10\beta_0 + 3\beta_{1.}$

$g(11) = g(2(5) + 1) = 3g(5) + 5\gamma + \beta_1$

$= 27\alpha + 20\gamma + 9\beta_0 + 4\beta_{1.}$

$g(12) = g(2(6) + 0) = 3g(6) + 6\gamma + \beta_0$

$= 27\alpha + 24\gamma + 4\beta_0 + 9\beta_1$

Evaluating "R" on some values (Continued)

$$g(13) = g(2(6) + 1) = 3g(6) + 6\gamma + \beta_1$$
$$= 27\alpha + 24\gamma + 3\beta_0 + 10\beta_{1.}$$
$$g(14) = g(2(7) + 0) = 3g(7) + 7\gamma + \beta_0$$
$$= 27\alpha + 25\gamma + 1\beta_0 + 12\beta_{1.}$$
$$g(15) = g(2(7) + 1) = 3g(7) + 7\gamma + \beta_1$$
$$= 27\alpha + 25\gamma + 13\beta_{1.}$$
$$g(16) = g(2(8) + 0) = 3g(8) + 8\gamma + \beta_0$$
$$= 81\alpha + 65\gamma + 40\beta_{0.}$$

STEP 2: Finding the General Form of the Closed Formula

n	*g (n)*
1	$\alpha + 0\gamma + 0\beta_0 + 0\beta_1$
2	$3\alpha + \gamma + \beta_0 + 0\beta_1$
3	$3\alpha + \gamma + 0\beta_0 + \beta_1$
4	$9\alpha + 5\gamma + 4\beta_0 + 0\beta_1$
5	$9\alpha + 5\gamma + 3\beta_0 + \beta_1$.
6	$9\alpha + 6\gamma + \beta_0 + 3\beta_1$.
7	$9\alpha + 6\gamma + 0\beta_0 + 4\beta_1$.
8	$27\alpha + 19\gamma + 13\beta_0 + 0\beta_1$.

n	$g(n)$
9	$27\alpha + 19\gamma + 12\beta_0 + \beta_1$
10	$27\alpha + 20\gamma + 10\beta_0 + 3\beta_1$
11	$27\alpha + 20\gamma + 9\beta_0 + 4\beta_1$
12	$27\alpha + 24\gamma + 4\beta_0 + 9\beta_1$
13	$27\alpha + 24\gamma + 3\beta_0 + 10\beta_1$
14	$27\alpha + 25\gamma + 1\beta_0 + 12\beta_1$
15	$27\alpha + 25\gamma + 0\beta_0 + 13\beta_1$

n	*g (n)*
16	$81\alpha + 65\gamma + 40\beta_0 + 0\beta_1$

So, now we can guess the general form of the closed formula !

The general form of the Closed Formula is …

$$g(n) = A(n)\,\alpha + B(n)\,\gamma + C(n)\,\beta_0 + D(n)\,\beta_1$$

(for all n element of Natural Numbers)

Another Observation

- For all $n \in N$ and $n \geq 1$, n can be written in the form $2^k + h$, where, 2^k is the highest power of 2, not exceeding n, and $0 \leq h < 2^k$. (Here $h \in N$). From our table we can observe that when $n = 2^k + h$, then the coefficient for α is 3^k.
- So, for example, let n=15, then $n = 2^3+7$, and the coefficient for α after computing g (n) is, $3^3 = 27$.
- [We need this observation in step 3, case 1.]

STEP 3: Find Exact Closed Formula Using Repertoire Method

CASE 1: $g(n) = A(n)$ for all $n \in N$

CASE 1

- Let us consider the case where,

$$\gamma = \beta_0 = \beta_1 = 0 \text{ and } \alpha = 1$$

- In this case our general Formula would become:

  $g(n) = A(n)\,1 + B(n)\,0 + C(n)\,0 + D(n)\,0$

  $g(n) = A(n)$

  for all $n \in \mathbb{N}$.

CASE 1 (Cont.)

So, now the recurrence relation becomes:

$$A(1) = \alpha$$

$$A(2n + j) = 3A(n)$$

for $j = 0,1$ and $n \geq 1$ and $n \in N$

CASE 1 (Cont.)

- Now we want to prove that:

  $A(2^k + h) = 3^k \alpha$ for all $k \geq 0$.

  (Here $0 \leq h < 2^k$ and $h \in \mathbb{N}$)

- It is easy to prove by induction that:

  $A(2^k + h) = 3^k \alpha$ for all $k \geq 0$

- So, we have fact 1 to be :

  $A(2^k + h) = 3^k \alpha$ ……… fact(1)

CASE 2: $g(n) = 1$ for all $n \in N$

CASE 2

- Now let us consider the constant function:

  $$g(n) = 1$$

  for all $n \in \mathbb{N}$.
- Now we use our "R" to compute, (if possible),
- α , γ , β_0 and β_1
- From $g(1) = \alpha$, we find that $g(1) = 1 = \alpha$.
- So $\alpha = 1$

CASE 2 (Cont.)

- Now we evaluate:

$$g(2n+0) = 3g(n) + \gamma n + \beta_0$$
$$1 = 3(1) + \gamma n + \beta_0$$
$$1 = 3 + \gamma n + \beta_0$$
$$1\text{-}3 = \gamma n + \beta_0$$
$$-2 = \gamma n + \beta_0$$

we get $\gamma = 0$ and $\beta_0 = -2$

CASE 2 (Cont.)

- Now we evaluate:

$$g(2n+1) = 3g(n) + \gamma n + \beta_1$$
$$1 = 3(1) + \gamma n + \beta_1$$
$$1 = 3 + \gamma n + \beta_1$$
$$1-3 = \gamma n + \beta_1$$
$$-2 = \gamma n + \beta_1$$

we get $\gamma = 0$ and $\beta_1 = -2$

CASE 2 (Cont.)

- Now by plugging the values that we got for α ,γ ,β_0 and β_1 into the "general form" of the Closed Formula we get:

  $g(n) = A(n)\,\alpha + B(n)\,\gamma + C(n)\,\beta_0 + D(n)\,\beta_1$

  $$1 = A(n) – 2C(n) – 2D(n)$$

  for all $n \in N$.

Thus Fact 2 is:

$$1 = A(n) – 2C(n) – 2D(n) \quad \text{for all } n \in N.$$

…………. Fact(2)

CASE 3: $g(n) = n$ for all $n \in \mathbb{N}$

CASE 3

- We now consider the case $g(n) = n$ for all $n \in N$
- We now plug in $g(n) = n$, in the recursive formula and see if we can compute α , γ , β_0 and β_1
- From $g(1) = \alpha$, we find that $g(1) = 1 = \alpha$. So $\alpha = 1$.

CASE 3 (Cont.)

- Now we evaluate:

$$g(2n+0) = 3g(n) + \gamma n + \beta_0$$

$$2n = 3n + \gamma n + \beta_0$$

$$0 = 3n-2n + \gamma n + \beta_0$$

$$0 = n + \gamma n + \beta_0$$

$$0 = (\gamma+1)n + \beta_0$$

we get $\gamma+1 = 0$ and $\beta_0 = 0$.

So, $\gamma = -1$ and $\beta_0 = 0$

CASE 3 (Cont.)

- Now we evaluate:

$$g(2n+1) = 3g(n) + \gamma n + \beta_1$$

$$2n+1 = 3n + \gamma n + \beta_1$$

$$1 = 3n-2n + \gamma n + \beta_1$$

$$1 = n + \gamma n + \beta_1$$

$$1 = (\gamma+1)n + \beta_1$$

we get $\gamma+1 = 0$ and $\beta_1 = 1$.

So, $\gamma = -1$ and $\beta_1 = 1$

CASE 3.

- So, we got the following values :

  $\alpha = 1,\ \gamma = -1,\ \beta_0 = 0$ and $\beta_1 = 1$

  By Plugging this into the “general form” of the Closed Formula we get the following fact :

  $$g(n) = A(n)\,\alpha + B(n)\,\gamma + C(n)\,\beta_0 + D(n)\,\beta_1$$

  $$n = A(n) - B(n) + D(n)$$

  for all n element of N

  ……………. fact (3)

Do we have enough facts?

- No, we do not have enough facts to be able to compute A(n), B(n), C(n) and D(n) so far. We have three equations, (namely, fact1, fact2 and fact3) but we have four unknowns. So, these facts are not enough for us to find A(n), B(n), C(n) and D(n). Therefore we need to find one more equation… So, let us try to find another equation………..

CASE 4: $g(n) = n^2$

for all $n \in N$

CASE 4

- Now let us consider the function:

  $$g(n) = n^2$$

  for all $n \in N$

- Now we use our "R" to compute, (if possible),
- α , γ , β_0 and β_1
- From $g(1) = \alpha$, we deduce that:

  $$g(1)=1^2=1=\alpha$$

- So, $\alpha=1$

CASE 4.(Cont.)

- Now let us look at the case:

$$g(2n+1) = 3g(n) + \gamma n + \beta_1$$
$$(2n+1)^2 = 3n^2 + \gamma n + \beta_1$$
$$4n^2 + 1 + 4n = 3n^2 + \gamma n + \beta_1$$
$$n^2 + 1 + 4n = \gamma n + \beta_1$$
$$n^2 + n(4 - \gamma) - \beta_1 = -1 \quad (\text{for all } n \in N)$$

This means that : $1 = 0$, ← CONTRADICTION

$4 - \gamma = 0$,

and $\beta_1 = 1$

CANNOT USE STRAIGHT FORWARD REPERTOIRE METHOD !

- In case 4 we reached a contradiction, therefore, we are not able to come up with at least 4 equations (facts), that we need in order to find A(n), B(n), C(n) and D(n).
- However, there is another way to find the closed formula for a recursive formula !
- We make use of ………

THE BINARY EXPANSION TECHNIQUE

Using Binary Representation For Finding Closed Formula.

- We are going to find a closed formula for R by using the binary representation of the input value to R.
- Before proceeding, there are some important observations to make.
- Let us review what R is:

$$g(1) = \alpha$$
$$g(2n+0) = 3g(n) + \gamma n + \beta_0$$
$$g(2n+1) = 3g(n) + \gamma n + \beta_1$$
$$\text{(for all } n \in N \text{ and } n \geq 1)$$

Observation (Cont.)

- Let “n” be a natural number greater than or equal to 1.
- Let us consider the case when the input value to R is of the form 2n+0 = 2n.
- Let the binary representation of 2n be:

$$(b_m, b_{m-1}, \ldots., b_1, b_0)_2$$

- This means $2n = 2^m b_m + \ldots + 2 b_1 + b_0$

---- (equation 1)

- Thus, we observe that $b_0 = 0$, because 2n is an even number.

Observation (Cont.)

- If we divide both sides of equation 1 by 2 then we get:

  $$n = 2^{m-1}b_m + \ldots + b_1$$

- Thus, $n = (b_m, b_{m-1}, \ldots, b_1)_2$

  ----- info (1)

Observation (Cont.)

- Now, Let us consider the case when the input value to R is of the form 2n+1.
- Let the binary representation of 2n+1 be:

  $$(b_m, b_{m-1}, \ldots., b_1, b_0)_2$$

  So, $2n+1 = 2^m b_m + \ldots + 2 b_1 + b_0$
- Since 2n+1 is an odd number $b_0 = 1$.
- So, now :

Observation (Cont.)

- $$2n+1 = 2^m b_m + \ldots + 2 b_1 + b_0$$
  $$2n+1 = 2^m b_m + \ldots + 2 b_1 + 1$$
  $$2n = 2^m b_m + \ldots + 2 b_1$$
  $$n = 2^{m-1} b_m + \ldots + b_1$$
- So, this means that $n = (b_m, \ldots, b_1)_2$
- Thus we can see that we would have the same binary representation for n, if either we are given the binary representation for 2n, or we are given the binary representation for 2n+1.
- Thus, when finding the exact closed formula we do not need to consider two cases for the binary representation for n.

Finding the Exact Closed Formula

- Now we can go ahead and find the exact closed formula for "R" by using the binary representation of the input value. (Input values can be 1 or 2n or 2n+1, where n is any natural number greater than or equal to 1)

- Let 2n and 2n+1 be represented as follows in binary:

$$(b_m, b_{m-1}, \ldots., b_1, b_0)_2$$

Finding Exact Closed Formula (Cont.)

$g((b_m, b_{m-1}, \ldots, b_1, b_0)_2)$

$= 3g((b_m, b_{m-1}, \ldots, b_1)_2) +$

$\gamma(b_m, b_{m-1}, \ldots, b_1)_2$

$+ \beta_{b0}$

NOTE:
This turns out to be just this without the right most bit.

(Applying formula)

$= 3(3g((b_m, b_{m-1}, \ldots, b_2)_2) +$

$\gamma(b_m, b_{m-1}, \ldots, b_2)_2 + \beta_{b1}) +$

$\gamma(b_m, b_{m-1}, \ldots, b_1)_2 \quad + \beta_{b0}$

OBSERVATION:

$\beta_{b_0} = \beta_1$ if $b_0 = 1$

And

$\beta_{b_0} = \beta_0$ if $b_0 = 0$

Similarly,

$\beta_{b_1} = \beta_1$ if $b_1 = 1$

And

$\beta_{b_1} = \beta_0$ if $b_1 = 0$

Finding Exact Closed Formula (Cont.)

$$g((b_m, b_{m-1}, \ldots., b_1, b_0)_2) = 3^2 g((b_m, b_{m-1}, \ldots., b_2)_2) + 3\gamma(b_m, b_{m-1}, \ldots., b_2)_2 + 3\beta_{b1} + \gamma(b_m, b_{m-1}, \ldots., b_1)_2 + \beta_{b0}$$

(just applying distributive property)

Finding Exact Closed Formula (Cont.)

$g((b_m, b_{m-1}, \ldots., b_1, b_0)_2)$

$$= 3^2(3g((b_m, b_{m-1}, \ldots., b_3)_2) + \gamma(b_m, b_{m-1}, \ldots., b_3)_2 + \beta_{b2}) + 3\gamma(b_m, b_{m-1}, \ldots., b_2)_2 + 3\beta_{b1} + \gamma(b_m, b_{m-1}, \ldots., b_1)_2 + \beta_{b0}$$

(just evaluating recursive formula again.)

Finding Exact Closed Formula (Cont.)

$g((b_m, b_{m-1}, \ldots., b_1, b_0)_2)$

$= 3^3 g((b_m, b_{m-1}, \ldots., b_3)_2) +$

$3^2 \gamma(b_m, b_{m-1}, \ldots., b_3)_2 + 3^2 \beta_{b2} +$

$3\gamma(b_m, b_{m-1}, \ldots., b_2)_2 + 3\beta_{b1} +$

$\gamma(b_m, b_{m-1}, \ldots., b_1)_2 + \beta_{b0}$

(Applying the distributive property.)

Finding Exact Closed Formula (Cont.)

$g((b_m, b_{m-1}, \ldots, b_1, b_0)_2)$

$= 3^3(3g((b_m, b_{m-1}, \ldots, b_4)_2) +$

$\gamma(b_m, b_{m-1}, \ldots, b_4)_2 + \beta_{b3}) +$

$3^2 \gamma(b_m, b_{m-1}, \ldots, b_3)_2 + 3^2 \beta_{b2} +$

$3\gamma(b_m, b_{m-1}, \ldots, b_2)_2 + 3\beta_{b1} +$

$\gamma(b_m, b_{m-1}, \ldots, b_1)_2 + \beta_{b0}$

(Applying the recursive formula again.)

So,…, we can see a pattern

$g\,((b_m, b_{m-1}, \ldots, b_1, b_0)_2\,)$

$= 3^m g((b_m)_2) +$
$\gamma(3^{m-1}(b_m)_2 + 3^{m-2}(b_m\, b_{m-1})_2 + \ldots +$
$3^0 (b_{m,\ldots,} b_1)_2\,) +$
$3^{m-1}\beta_{\mathbf{bm-1}} + 3^{m-2}\beta_{\mathbf{bm-2}} + \ldots + 3^1\beta_{\mathbf{b1}} +$
$3^0\beta_{\mathbf{b0}}$

------- eq(2)

What is b_m?

It was mentioned before that 2n and 2n+1 can be represented as

$$(b_m, b_{m-1}, \ldots., b_1, b_0)_2$$

Input value “1” is represented by only one bit “b_m.” In this case $b_m = 1$.

Also, the only time when b_m is 0, is when the decimal number itself is 0. Any natural number is going to have $b_m = 1$.

What is b_m?

Since the input to R are all natural numbers, therefore, in our computation :

$b_m = 1$.

- So, $g(b_m) = g(1) = \alpha$
- Now our eq(2) becomes:

Substituting "α"

$$g((b_m, b_{m-1}, \ldots, b_1, b_0)_2)$$

$$= 3^m \alpha +$$

$$\gamma(3^{m-1} (b_m)_2 + 3^{m-2} (b_m b_{m-1})_2 + \ldots + 3^0 (b_{m,\ldots,} b_1)_2) +$$

$$3^{m-1} \beta_{b_{m-1}} + 3^{m-2} \beta_{b_{m-2}} + \ldots + 3^1 \beta_{b_1} + 3^0 \beta_{b_0}$$

Final Answer (The Closed Formula)

$g(1) = 1$

For any input value greater than 1, which can be represented as $(b_m, b_{m-1}, \ldots., b_1, b_0)_2$:

$$g\left((b_m, b_{m-1}, \ldots., b_1, b_0)_2\right) = 3^m \alpha + \gamma \sum 3^{m-i}(b_m \ldots\ldots b_{m-i+1})_2 + \sum 3^{m-i}\beta_{b_{m-i}}$$

(Limits of both summations: from i=1 to m)

EXAMPLE

Compute $g(n)$ using the closed formula, when $n=8$.

EXAMPLE

- So, we know that $n = 8$
- We first find the binary representation for 8. So, $n = 8 = (1000)_2$
- Over here $m = 3$.
- $b_0 = 0$, $b_1=0$, $b_2= 0$ and $b_3= b_m = 1$
- We know that our closed formula is of the following form:

EXAMPLE (Cont.)

- $g\,(b_m, b_{m-1}, \ldots, b_1, b_0)_2 =$

$$3^m \alpha + \gamma \sum_{i=1}^{m} 3^{m-i}(b_m \ldots\ldots b_{m-i+1}) + \sum_{i=1}^{m} 3^{m-i}\beta_{b_{m-i}}$$

- $g((1000)_2) =$

$$3^3 \alpha + \gamma \sum_{i=1}^{3} 3^{3-i}(b_3 \ldots\ldots b_{3-i+1}) + \sum_{i=1}^{3} 3^{3-i}\beta_{b_{3-i}}$$

EXAMPLE (Cont.)

- $g((1000)_2) =$
  $= 3^3 \alpha +$

  $\gamma(3^{3-1}(b_3 \ldots\ldots b_{3-1+1})_2 +$

  $3^{3-2}(b_3 \ldots\ b_{3-2+1})_2 +$

  $3^{3-3}(b_3 \ldots\ldots b_{3-3+1})_2) +$

  $3^{3-1}\beta_{b_{3-1}} + 3^{3-2}\beta_{b_{3-2}} + 3^{3-3}\beta_{b_{3-3}}$

EXAMPLE (Cont.)

- $g((1000)_2) =$
  $= 3^3 \alpha +$

  $\gamma(3^2 (b_3)_2 + 3^1 (b_3 b_2)_2 +$

  $3^0 (b_3 b_2 b_1)_2) +$

  $3^2 \beta_{b_2} + 3^1 \beta_{b1} + 3^0 \beta_{b_0}$

EXAMPLE (Cont.)

- $g((1000)_2) =$
  $= 3^3 \alpha +$

  $\gamma(3^2 (1)_2 + 3^1 (10)_2 +$

  $3^0 (100)_2) +$

  $3^2 \beta_0 + 3^1 \beta_0 + 3^0 \beta_0$

EXAMPLE (Cont.)

- $g((1000)_2) =$

  $27\alpha +$

  $\gamma(9 + (3)(2) + (1)(4))$

  $+ 9\beta_0 + 3\beta_0 + \beta_0$

- $g((1000)_2) = 27\alpha + 19\gamma + 13\beta_0$

EXAMPLE (Cont.)

- Thus, we got :

$$g(8) = 27\alpha + 19\gamma + 13\beta_0$$

- If we compute g(8) by directly using the Recursive formula we get the same result.

- Please note that the recursive computation of g(8) is shown on slides # 8 and #7