Proof of Partition Theorem

Spec($\sqrt{2}$) and Spec($2 + \sqrt{2}$)

Form a Partition of N-{0} = Z+

Method : Prove General Theorem of which our theorem is a particular case

SPECTRUM PARTITION THEOREM

Let $\alpha, \beta > 0$, $\alpha, \beta \in R-Q$

Be such that $1/\alpha + 1/\beta = 1$

Then the sets

$A = \{ \lfloor \alpha n \rfloor : n = 1,2,3,....\} = spec(\alpha)$

$A = \{ \lfloor \beta n \rfloor : n = 1,2,3,....\} = spec(\beta)$

Form a Partition of Z+ = N – {0}

i.e. A≠ϕ, B≠ϕ

A∩B = ϕ

AUB = Z+

Proof of Partition Theorem (Special Case)

$\alpha = \sqrt{2}$, $\beta = 2 + \sqrt{2}$

We get :

Spec($\sqrt{2}$) and Spec($2 + \sqrt{2}$)

Form a Partition of N-{0} = Z+

Proof

1. The $\lfloor \alpha \rfloor \in A$, $\lfloor \beta \rfloor \in B$
2. $A \cap B = \phi$

Proof by contradiction

Suppose that $A \cap B \neq \phi$ i.e.

i.e. There is $k \in Z+$ such that $k \in A$, $k \in B$

Iff there are $i, j \in Z+$ such that

$\lfloor \alpha i \rfloor = k$ $\lfloor \beta j \rfloor = k$ **i.e.**

$k \leq \alpha i < k+1$

$k \leq \beta j < k+1$

But $\alpha, \beta \in R-Q$,

So αi, βj cant be integers, so $\leq$ cant hold, so we get

$k < \alpha i < k+1$

$k < \beta j < k+1$

$k/\alpha < i < (k+1)/\alpha$

$k/\beta < j < (k+1)/\beta$

On adding them

$k/\alpha + k/\beta < i + j < (k+1)/\alpha + (k+1)/\beta$

$k(1/\alpha + 1/\beta) < i + j < (k+1)(1/\alpha + 1/\beta)$

We know that $1/\alpha + 1/\beta = 1$

=> $k < i+j < k+1$

Contradiction!!!

$i,j,k \in Z+$

No integer between k, k+1!

We proved $A \cap B = \phi$

Now we want to prove that AUB = Z+

Assume AUB ≠ Z+

i.e. Exists $k \in Z+$, such that $k \notin AUB$

i.e. $k \notin A$ and $k \notin B$

$k \notin A$ iff for all $x \in Z+$ $\quad k \neq \lfloor \alpha n \rfloor$ ---------- 1

$k \notin B$ iff for all $x \in Z+$ $\quad k \neq \lfloor \beta n \rfloor$ ---------- 2

This means that there exist i_0, j_0 such that

$\lfloor \alpha\, i_0 \rfloor < k$ and $\lfloor \alpha\,(i_0+1) \rfloor > k$ and same holds for β

i.e.

(1a) $\alpha\, i_0 < k$ & $\alpha\,(i_0+1) > k+1$

(2a) $\beta\, i_0 < k$ & $\beta\,(i_0+1) > k+1$

(These cant be = k+1 as β,α Є R-Q and k+1 Є Z+)

$\lfloor \beta\, i_0 \rfloor < k$ and $\lfloor \beta\,(i_0+1) \rfloor > k$

1a when rewritten

$\alpha < k/ i_0$ & $\alpha > (k+1)/ (i_0+1)$

=> $1/\alpha > i_0/ k$ & $1/\alpha < (i_0+1)/ (k+1)$

Or $i_0/ k < 1/\alpha < (i_0+1)/ (k+1)$ |||ly for β we get

$j_0/ k < 1/ \beta < (j_0+1)/ (k+1)$

Adding the above two equations and using $1/\alpha + 1/\beta = 1$ We get

$(i_o+j_o)/k < 1 < (i_o+j_o+2)/(k+1)$

$\Rightarrow (i_o+j_o)/k < 1$ and $1 < (i_o+j_o+2)/(k+1)$

$\Rightarrow i_o+j_o < k$ and $k < i_o+j_o+1$

$\Rightarrow i_o+j_o < k < i_o+j_o+1 \quad k, i_o, j_o \in Z+$

Contradiction: $n < k < n+1$