Infinite Sums (Series) - Examples

Example 1

Prove that \( \sum_{n=1}^{\infty} \frac{c^n}{n!} \) converges for \( c > 0 \)

Hint: Use D'Alembert

Proof:

\[
\frac{a_{n+1}}{a_n} = \frac{c^{n+1}}{c^n} \times \frac{n!}{n+1!} = \frac{c}{n+1}
\]

\[
\lim_{n \to \infty} \frac{a_{n+1}}{a_n} = \lim_{n \to \infty} \frac{c}{n+1} = 0, \quad < 1
\]

By D'Alambert's Criterion, Any series with all its terms being positive real numbers i.e

\[ \sum_{n=1}^{\infty} a_n \text{ converges if } \lim_{n \to \infty} \frac{a_{n+1}}{a_n} < 1 \]

Hence, \( \sum_{n=1}^{\infty} \frac{c^n}{n!} \) converges
Example 2

Prove that \( \sum_{n=1}^{\infty} \frac{n!}{n^n} \) converges

Proof:

\[
a_n = \frac{n!}{n^n}
\]

\[
a_{n+1} = \frac{n!(n+1)}{(n+1)^{n+1}}
\]

\[
\frac{a_{n+1}}{a_n} = \frac{n!(n+1)}{(n+1)^{n+1}} \cdot \frac{n^n}{n^n} = (n+1) \cdot \frac{n^n}{(n+1)^{n+1}}
\]

\[
(n+1)^{n+1} = (n+1)^n \cdot (n+1)
\]

\[
\frac{a_{n+1}}{a_n} = \frac{(n+1) \cdot n^n}{(n+1)^n \cdot (n+1)} = \left(\frac{n}{n+1}\right)^n = \frac{1}{\left(1 + \frac{1}{n}\right)^n}
\]

\[
\lim_{n \to \infty} \frac{a_{n+1}}{a_n} = \lim_{n \to \infty} \frac{1}{\left(1 + \frac{1}{n}\right)^n} = \frac{1}{e} < 1
\]

By D’Alambert’s Criterion, Any series with all its terms being positive real numbers i.e

\[
\sum_{n=1}^{\infty} a_n \text{ converges if } \lim_{n \to \infty} \frac{a_{n+1}}{a_n} < 1
\]

Hence, \( \sum_{n=1}^{\infty} \frac{n!}{n^n} \) converges
Exercise 1:

Prove that \[ \lim_{n \to \infty} \frac{c^n}{n!} = 0 \quad c > 0 \]

Solution:

We have proved in Example 1, 
\[ \sum_{n=1}^{\infty} \frac{c^n}{n!} \text{ converges for } c > 0 \]

By Theorem 1, we have 
\[ \text{If } \sum_{n=1}^{\infty} a_n \text{ converges then } \lim_{n \to \infty} a_n = 0 \]

Hence from the proof of Example 1 and Theorem 1, we have proved that 
\[ \lim_{n \to \infty} \frac{c^n}{n!} = 0 \text{ for } c > 0 \]

Exercise 2

Prove that \[ \lim_{n \to \infty} \frac{n!}{n^n} = 0 \quad \text{hint: Complicate it!} \]

Proof:

By Example 2, we have proved that 
\[ \sum_{n=1}^{\infty} \frac{n!}{n^n} \text{ converges} \]

By Theorem 1, we have 
\[ \text{if } \sum_{n=1}^{\infty} a_n \text{ converges then, } \lim_{n \to \infty} a_n = 0 \]

Hence, 
\[ \lim_{n \to \infty} \frac{n!}{n^n} = 0 \]
Example 3

Prove that Harmonic Series \( H = \sum_{n=1}^{\infty} \frac{1}{n} \) does not react on D’Alembert Criterium

Proof:

By D’Alembert’s Criterium, we need to prove, \( \lim_{n \to \infty} \frac{a_{n+1}}{a_n} < 1 \)

\[
\frac{a_{n+1}}{a_n} = \frac{1}{n+1} \cdot \frac{n}{1}
\]

\[
= \frac{n}{n+1}
\]

\[
= \frac{1}{1 + \frac{1}{n}}
\]

\( \lim_{n \to \infty} \frac{a_{n+1}}{a_n} = 1 \) contradictory to D’Alembert’s criterium.

Hence, the Harmonic series does not react on D’Alembert’s Criterium.

Example 4

Prove that \( \lim_{n \to \infty} \frac{e^n}{n!} = 0 \) and \( \lim_{n \to \infty} \frac{n!}{n^n} = 0 \)

Proof:

By Example 1, we have proved that

\[ \sum_{n=1}^{\infty} \frac{e^n}{n!} \] converges

By Example 2, we have proved that
\[
\sum_{n=1}^{\infty} \frac{n!}{n^n} \text{ converges}
\]

By Theorem 1, we have

\[
\text{if } \sum_{n=1}^{\infty} a_n \text{ converges then, } \lim_{n \to \infty} a_n = 0
\]

Hence,

\[
\lim_{n \to \infty} \frac{c^n}{n!} = 0, \quad \lim_{n \to \infty} \frac{n!}{n^n} = 0
\]

**Example 5**

*We know that Harmonic Series \( \sum_{n=1}^{\infty} \frac{1}{n} \) diverges. Use this information and Cauchy Criteria to prove that, \( \lim_{n \to \infty} \sqrt[n]{n} = 1 \)

**Proof:**

We observe the sequence \( a_n = \sqrt[n]{n} \) for large \( n \), decreasing and \( a_n > 1 \), hence \( \lim_{n \to \infty} a_n \) exists and \( \lim_{n \to \infty} \sqrt[n]{n} \geq 1 \).

Assume now that \( \lim_{n \to \infty} \sqrt[n]{n} > 1 \), we get \( \lim_{n \to \infty} \sqrt[n]{\frac{1}{n}} < 1 \)

*By Cauchy Criterion if \( \lim_{n \to \infty} \sqrt[n]{a_n} < 1 \) then, \( \sum_{n=1}^{\infty} a_n \) converges for \( a_n \geq 0, a_n \in \mathbb{R} \)

*Hence, \( \sum_{n=1}^{\infty} \frac{1}{n} \) converges

This is a **contradiction**, as we know that the Harmonic Series,

\[
\sum_{n=1}^{\infty} \frac{1}{n} \text{ diverges}
\]

Hence,

\[
\lim_{n \to \infty} \sqrt[n]{n} = 1
\]
Example 6

Prove that \( \sum_{n=1}^{\infty} \frac{|x(x-1)\ldots(x-n+1)|}{n!} c^n \) converges for \( 0 < c < 1 \), \( x \in \mathbb{R} \).

Proof:

We evaluate

\[
\frac{a_{n+1}}{a_n} = \frac{|x(x-1)\ldots(x-n)|.c^n.c}{n!(n+1)} \cdot \frac{n!}{|x(x-1)\ldots(x-n+1)|.c^n} \]

\[
= \frac{|x-n|}{n+1} \cdot c
\]

\[
= \frac{|x-n|}{n+1} \cdot c
\]

\[
\lim_{n \to \infty} \frac{a_{n+1}}{a_n} = c
\]

\[
\lim_{n \to \infty} \frac{a_{n+1}}{a_n} \text{ converges for } 0 < c < 1
\]

Hence,

\[
\sum_{n=1}^{\infty} \frac{|x(x-1)\ldots(x-n+1)|}{n!} c^n \text{ converges for } 0 < c < 1, \quad x \in \mathbb{R}
\]
Example 7:

\[ \text{Prove that } \lim_{n \to \infty} \frac{|x(x-1)\cdots(x-n+1)|}{n!} c^n = 0 \quad 0 < |c| < 1 \]

Solution:

By Example 6, the series is proved to be convergent for \(0 < |c| < 1\)

By Theorem 1, we have

\[ \text{If } \sum_{n=1}^{\infty} a_n \text{ converges then } \lim_{n \to \infty} a_n = 0 \]

Hence proved.

Absolute and Conditional Convergence

Definition - Absolute Convergence

The series \( \sum_{n=1}^{\infty} a_n \) converge absolutely iff \( \sum_{n=1}^{\infty} |a_n| \) converges.

Definition - Conditional Convergence

The series \( \sum_{n=1}^{\infty} a_n \) converge conditionally iff \( \sum_{n=1}^{\infty} |a_n| \) converges, but not absolutely

i.e \( \sum_{n=1}^{\infty} a_n \) converges and \( \sum_{n=1}^{\infty} |a_n| \) does not converge.

Theorem 10

If \( \sum_{n=1}^{\infty} a_n \) converges absolutely, then it converges

Moreover, \( \sum_{n=1}^{\infty} |a_n| \leq \sum_{n=1}^{\infty} |a_n| \)
Example 8

Geometric series \( \sum_{n=0}^{\infty} aq^n, |q| < 1 \) converges absolutely because \( \sum_{n=1}^{\infty} |aq^n| \) converges.

Example 9

\( \sum_{n=0}^{\infty} \frac{x^n}{n!} \), converges absolutely for all \( x \) (we proved in example 1 converges for \( c > 0 \)

i.e \( |x| \sum_{n=0}^{\infty} \frac{|x|^n}{n!} = e^x \)

Example 10

The ENHarmonic series, \( \sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n} \) converges conditionally because it CONVERGES

and \( |(-1)^{n+1} \frac{1}{n}| = \frac{1}{n} = |a_n| \), so \( \sum_{n=1}^{\infty} |a_n| \) diverges.

Finite and Infinite Commutativity

We know that finite summation is commutative

\( \sum_{k=1}^{n} a_k = \sum_{k=1}^{n} a_{ik} \), where \( a_{ik} \) is any permutation of \( a_1...a_n \)

The Commutativity fails

For some Infinite Sums as we showed for example evaluating

\( \sum_{k>0} (-1)^k = \sum_{k=0}^{\infty} (-1)^k = 1 - 1 + 1 - 1 + 1 \ldots \)

By grouping (permutating) the sum factors in 2 different ways:

1. \( \sum_{k=0}^{\infty} (-1)^k = (1-1)+(1-1)+\ldots = 0 \)
2. \( \sum_{k=1}^{\infty} (-1)^k = 1-(1-1)-(1-1)\ldots = 1 \)

Question: When for which infinite sums commutativity holds and for which fails
Let $a_n$ be a sequence, $a_{m_k}$ is a sequence of permutations of $a_n$

**Definition**

A Permutation of a set $A$ is any function

$$f : A \xrightarrow{1-1 \text{ onto}} A$$

where $A$ is any cardinality

In particular

$$f : N \xrightarrow{1-1 \text{ onto}} N$$

is a permutation of natural numbers and we denote $f(n)=m_n$

Given $\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \ldots$

Infinite Sum $\sum_{k=1}^{\infty} a_{m_k} = a_{m_1} + a_{m_2} + \ldots$ is a permutation of natural numbers.

**Theorem 11**

Every $\sum_{n=1}^{\infty} a_n$ absolutely convergent infinite sum is commutative i.e

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} a_{m_n}$$

for any permutation $m_1 \ m_2 \ldots \ m_n \ldots$ of natural numbers

NOT TRUE for any convergent sum: we can get from ANHARMONIC series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$$

, permutation that converges or diverges to $\infty$

**Theorem 12 : Riemann Theorem**

For any conditionally convergent infinite sum, we can transform it by permutation of its factors into a sum that diverges, or to a sum that converges to any limit (finite or infinite)