QUESTION 1
Part 1 Prove that
\[ \sum_{k=2}^{n} \frac{(-1)^k}{2k-1} = -\sum_{k=1}^{n-1} \frac{(-1)^k}{2k+1} \]
Part 2 Use partial fractions and Part 1 result (must use it!) to evaluate the sum
\[ S = \sum_{k=1}^{n} \frac{(-1)^k}{4k^2 - 1} \]

QUESTION 2 Give a direct proof from proper properties (use the list) of the following.
For all \( x \in \mathbb{R}, x > 0 \)
\[ \lfloor \sqrt{\lfloor x \rfloor} \rfloor = \lfloor \sqrt{x} \rfloor \]

QUESTION 3
1. Prove that the series \( \sum_{n=1}^{\infty} \frac{1}{(n + 1)^2} \) does not react on D’Alambert’s Criterium
2. Prove that the series \( \sum_{n=1}^{\infty} \frac{n!}{n^n} \) converges.

QUESTION 4 Solve the recurrence: for \( n > 0 \)
\[ a_0 = 1, \quad a_n = a_{n-1} + \lfloor \sqrt{a_{n-1}} \rfloor, \quad \text{for } n > 0 \]
Hint assume first that \( a_n = m^2 \) for certain \( m \in \mathbb{Z} \) and find formulas for \( a_{n+2k+1} \) and \( a_{n+2k+2} \).

QUESTION 5 Prove the following.
1. Let \( m, n, k \in \mathbb{Z} + \{-0\}. \)
   If \( k \mid mn \) and \( k \perp m \) (it means \( k, m \) are relatively prime), THEN \( k \mid n \).
2. When a number is relatively prime to each of several numbers, it is relatively prime to their product.

QUESTION 6 Write a proof of the following:
\( \text{spec}(\sqrt{2}) \) and \( \text{spec}(2 + \sqrt{2}) \) are disjoint sets.

QUESTION 7 Find the sum of all multiples of \( x \) in the closed interval \([\alpha...\beta]\), when \( x > 0 \).
Justify methods used in each step of your calculation.

QUESTION 8 Denote by \( N(\alpha, n) \) the number of elements in the \( \text{Spec}(\alpha) \) that are \( \leq n \), i.e.
\[ N(\alpha, n) = | \{ m \in \text{Spec}(\alpha) : m \leq n \} |. \]
Write a detailed proof of

\[ N(\alpha, n) = \left\lceil \frac{n+1}{\alpha} \right\rceil - 1. \]

No credit without each step explanations.

**QUESTION 9** Show that the nth element of the sequence:

\[ 1, 2, 2, 3, 3, 4, 4, 4, 5, 5, 5, 5, \ldots \]

is \[ \lfloor \sqrt{2n} + \frac{1}{2} \rfloor \].

**Hint:** Let \( P(x) \) represent the position of the last occurrence of \( x \) in the sequence.
Use the fact that \( P(x) = \frac{x(x+1)}{2} \).
Let the nth element be \( m \). You need to find \( m \).

**QUESTION 10** Prove that

\[ \binom{m}{n} \binom{n}{k} = \binom{m}{k} \binom{n-k}{m-k} \]

holds for all \( m, k \in \mathbb{Z} \) and \( x \in \mathbb{R} \). Consider all cases and Polynomial argument. No credit without all cases and pol. argument!

**QUESTION 11** Prove the Hexagon property \((n, k \in \mathbb{N})\)

\[ \binom{n-1}{k-1} \binom{n}{k+1} \binom{n+1}{k} = \binom{n-1}{k} \binom{n+1}{k+1} \binom{n}{k-1} \]