cse547, math547
DISCRETE MATHEMATICS

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LECTURE 4a

CHAPTER 1, Problem 20
SOLUTION

Problem

Use the repertoire method to solve the general **five-parameter recurrence** RF

Solve means FIND the closed formula CF equivalent to RF

$$\begin{aligned} h(1) &= \alpha; \\ h(2n+0) &= 4h(n) + \gamma_0 n + \beta_0; \\ h(2n+1) &= 4h(n) + \gamma_1 n + \beta_1, \quad \text{for all } n \geq 1. \end{aligned}$$

General Form of CF

Our RF for h is a FIVE parameters function and it is a **generalization** of the General Josephus GJ function f considered before

So we guess that now the **general form** of the CF is also a generalization of the one we already proved for GJ , i.e.

General form of CF is

$$h(n) = \alpha A(n) + \gamma_0 B(n) + \gamma_1 C(n) + \beta_0 D(n) + \beta_1 E(n)$$

The **Problem 20** asks us to use repertoire method to prove that CF is **equivalent** to RF

Thinking Time

Solution requires a system of **5 equations** on A(n), B(n), C(n), D(n), E(n) and accordingly a **5 repertoire functions**!

Let's THINK a bit before we embark on quite complicated calculations and without certainty that they would succeed (look at thesolution to the **Problem 16**)

First : we observe that when when $\gamma_0 = \gamma_1 = 0$, we get that h becomes for Generalize Josephus function f below for k = 4:

$f(1) = \alpha, \quad f(2n + j) = \; kf(n) + \beta_j,$

where $k \geq 2, \; j = 0, 1$ and $n \geq 0$

It seems **worth to examine** the case $\gamma_0 = \gamma_1 = 0$ **first**

GJ f Closed Formula Solution

We **proved** that GJ function f has a relaxed k-representation closed formula

$$\mathbf{f((1, b_{m-1}, ...b_1, b_0)_2)} = (\alpha, \beta_{\mathbf{b_{m-1}}}, ...\beta_{\mathbf{b_0}})_{\mathbf{k}}$$

where β_{b_j} are defined by

$$\beta_{b_j} = \begin{cases} \beta_0 & b_j = 0 \\ \beta_1 & b_j = 1 \end{cases} \quad ; \quad j = 0, ..., m-1,$$

for the **relaxed k- radix representation** defined as

$$(\alpha, \beta_{\mathbf{b_{m-1}}}, ..., \beta_{\mathbf{b_0}})_{\mathbf{k}} = \alpha \mathbf{k^m} + \mathbf{k^{m-1}} \beta_{\mathbf{m-1}} + ... + \beta_{\mathbf{b_0}}$$

Special Case of h

Consider now a special case of our h, when $\gamma_0 = \gamma_1 = 0$

We know that it now has a relaxed 4 - representation closed formula

$$h((1, b_{m-1}, ...b_1, b_0)_2) = (\alpha, \beta_{b_{m-1}}, ...\beta_{b_0})_4$$

It means that we get

Fact 0 For any $n = (1, b_{m-1}, ...b_1, b_0)_2$,

$$h(n) = (\alpha, \beta_{b_{m-1}}, ...\beta_{b_0})_4$$

Observe that our general form of CF in this case becomes

$$h(n) = \alpha A(n) + \beta_0 D(n) + \beta_1 E(n)$$

We must have h(n) = h(n), for all n, so from this and **Fact 0** we get the following equation 1 (stated as Fact 1)

Equation 1

Fact 1 For any $n = (1, b_{m-1}, ...b_1, b_0)_2$,

$$\alpha A(n) + \beta_0 D(n) + \beta_1 E(n) = (\alpha, \beta_{b_{m-1}}, ...\beta_{b_0})_4$$

This provides us with the Equation 1 for finding our general form of CF

Next Observation

Observe that A(n) in the Original Josephus was given (and proved to be) by a formula

$A(n) = 2^k$, for all $n = 2^k + \ell,\ \ 0 \leq \ell < 2^k$

So we wonder if we could have a **similar solution** for our A(n)

Special Case of h

We evaluate now few initial values for h in case $\gamma_0 = \gamma_1 = 0$

$$\begin{aligned} h(1) &= \alpha; \\ h(2) &= h(2(1)+0) = 4h(1) + \beta_0 \\ &= 4\alpha + \beta_0; \end{aligned}$$

$$\begin{aligned} h(3) &= h(2(1)+1) = 4h(1) + \beta_1 \\ &= 4\alpha + \beta_1; \end{aligned}$$

$$\begin{aligned} h(4) &= h(2(2)+0) = 4h(2) + \beta_0 \\ &= 16\alpha + 5\beta_0; \end{aligned}$$

Equation 2

It is pretty obvious that we do have a similar formula for A(n) as on the Original Josephus OJ

We write it as our **Fact 2** and get our Equation 2.

Fact 2

For all $n = 2^k + \ell, \;\; 0 \leq \ell < 2^k, \; n \in N - \{0\}$

$$A(n) = 4^k$$

The proof is almost identical to the one in the OJ, we re-write is here for our case as an exercise.

Reminder

We investigate the case when $\gamma_0 = \gamma_1 = 0$, i.e. now our formulas are

RF: $h(1) = \alpha, \quad h(2n + j) = 4h(n) + \beta_j$

where $j = 0, 1$ and $n \geq 0$ and the closed formula is

CF: $h(n) = \alpha A(n) + \beta_0 D(n) + \beta_1 E(n)$

Proof of the Equation 2

Consider now a further case $\beta_0 = \beta_1 = 0$, and $\alpha = 1$, i.e.

RF : $h(1) = 1, \quad h(2n) = 4h(n), \quad h(2n+1) = 4h(n)$

and CF : $h(n) = A(n)$

We use $h(n) = A(n)$ and re-write RF in terms of A(n)

RA : $A(1) = 1, \quad A(2n) = 4A(n), \quad A(2n+1) = 4A(n)$

Fact Closed formula CAR for AR is:

CA: $A(n) = A(2^k + \ell) = 4^k, \quad 0 \leq \ell < 2^k$

Observe that this **Fact** is equivalent to our **Fact 2**, i.e. to validity of the Equation 2, so we are now proving

Fact 2 for all $n = 2^k + \ell, \ 0 \leq \ell < 2^k$

$$A(n) = 4^k$$

Proof of the Equation 2

Proof by induction on k

Base case: k=0 i.e $n = 2^0 + \ell$, $0 \le \ell < 1$, hence n = 1 and RA: A(1) = 1, and CA: A(1) = 4^0 = 1, so we have RA = CA

Inductive Assumption

$A(2^{k-1} + \ell) = A(2^{k-1} + \ell) = 4^{k-1}$, for $0 \le \ell < 2^{k-1}$

Inductive Thesis

$A(2^k + l) = A(2^k + l) = 4^k$, for $0 \le l < 2^k$

Two cases: $n \in even$, $n \in odd$

C1: $n \in even$

n := 2n, and we have $2^k + \ell = 2n$ iff $\ell \in even$

Proof of the Equation 2

We evaluate n as follows

$2n = 2^k + \ell, \quad n = 2^{k-1} + \frac{\ell}{2}$

We use n in the **inductive step**

Observe that the **correctness** of using $\frac{\ell}{2}$ follows from that fact that $\ell \in even$, so $\frac{\ell}{2} \in N$ and it can be proved formally like on the previous slides

Inductive Proof

$A(2n) =^{reprn} A(2^k + \ell) =^{n-eval} 4A(2^{k-1} + \frac{\ell}{2}) =^{ind}$
$4 * 4^{k-1} = 4^k$

Proof of the Equation 2

C2: $n \in odd$

n:= 2n+1, and we have $2^k + \ell = 2n + 1$ iff $\ell \in odd$

We evaluate n as follows

$2n + 1 = 2^k + \ell$, $n = 2^{k-1} + \frac{\ell-1}{2}$

We use n in the **inductive step**

Observe that the correctness of using $\frac{\ell-1}{2}$ follows from that fact that $\ell \in odd$, so $\frac{\ell-1}{2} \in N$

Inductive Proof

$A(2n+1) =^{reprn} A(2^k + \ell) =^{n-eval} 4A(2^{k-1} + \frac{\ell-1}{2}) =^{ind}$
$4 * 4^{k-1} = 4^k$

It ends the proof of the **Fact 2:** $A(n) = 4^k$

Repertoire Method

We return now to out **original functions**:

RF: $h(1) = \alpha, h(2n) = 4h(n) + \gamma_0 n + \beta_0,$
$h(2n+1) = 4h(n) + \gamma_1 n + \beta_1,$

CF: $h(n) = \alpha A(n) + \gamma_0 B(n) + \gamma_1 C(n) + \beta_0 D(n) + \beta_1 E(n)$

We have already developed two equations (as stated in **Facts 1, 2**) so we need now to consider only **3 repertoire functions** to obtain 5 equations we need to solve the problem

Repertoire Function 1

Consider a **first repertoire** function : h(n) = 1, for all $n \in N - \{0\}$

We have h(n) = 1, h(1)= α, so we get $\alpha = 1$ and we evaluate

$h(2n) = 4h(n) + \gamma_0 n + \beta_0$	$h(2n+1) = 4h(n) + \gamma_1 n + \beta_1$;
$1 = 4 + \gamma_0 n + \beta_0$	$1 = 4 + \gamma_1 n + \beta_1$
$0 = 3 + \gamma_0 n + \beta_0$	$0 = 3 + \gamma_1 n + \beta_1$

We get $\gamma_0 = \gamma_1 = 0, \quad \beta_0 = \beta_1 = -3$

Solution 1:

$$\alpha = 1, \quad \gamma_0 = \gamma_1 = 0, \quad \beta_0 = \beta_1 = -3$$

Equation 3

CF: $\quad h(n) = \alpha A(n) + \gamma_0 B(n) + \gamma_1 C(n) + \beta_0 D(n) + \beta_1 E(n)$

We evaluate CF for our **Solution 1**:

$\alpha = 1, \gamma_0 = \gamma_1 = 0, \quad \beta_0 = \beta_1 = -3$ and get

CF = RF iff the following holds

Fact 3 For all $n \in N - \{0\}$,

$$A(n) - 3D(n) - 3E(n) = 1$$

This is our equation 3

Repertoire Function 2

Consider a **repertoire function 2**: h(n) = n, for all $n \in N - \{0\}$

h(1) = α, h(1) = 1 and h(n)=h(n), hence $\alpha = 1$

$$h(2n) = 4h(n) + \gamma_0 n + \beta_0$$
$$2n = 4n + \gamma_0 n + \beta_0$$
$$0 = (\gamma_0 + 2)n + \beta_0$$

$$h(2n+1) = 4h(n) + \gamma_1 n + \beta_1;$$
$$2n + 1 = 4n + \gamma_1 n + \beta_1$$
$$0 = (\gamma_1 + 2)n + (\beta_1 - 1)$$

We get $\gamma_0 = \gamma_1 = -2,\ \beta_0 = 0,\ \beta_1 = 1$ and

Solution 2:

$$\alpha = 1, \quad \gamma_0 = \gamma_1 = -2,\ \beta_0 = 0,\ \beta_1 = 1$$

Equation 4

CF: $h(n) = \alpha A(n) + \gamma_0 B(n) + \gamma_1 C(n) + \beta_0 D(n) + \beta_1 E(n)$

We evaluate CF for the **Solution 2:**

$\alpha = 1, \gamma_0 = \gamma_1 = -2,\ \beta_0 = 0,\ \beta_1 = 1$ and get

CF = RF iff the following holds

Fact 4 For all $n \in N - \{0\}$

$$A(n) - 2B(n) - 2C(n) + E(n) = n$$

This is our equation 4

Repertoire Function 3

Consider a **repertoire function 3**: $h(n) = n^2$, for all $n \in N$

h(1) = α, h(1) = 1, hence $\alpha = 1$

$$h(2n+0) = 4h(n) + \gamma_0 n + \beta_0$$
$$(2n)^2 = 4n^2 + \gamma_0 n + \beta_0$$
$$\cancel{4}n^2 = \cancel{4}n^2 + \gamma_0 n + \beta_0$$
$$0 = \gamma_0 n + \beta_0$$

$$h(2n+1) = 4h(n) + \gamma_1 n + \beta_1;$$
$$(2n+1)^2 = 4n^2 + \gamma_1 n + \beta_1$$
$$4n^2 + 4n + 1 = 4n^2 + \gamma_1 n + \beta_1$$
$$0 = (\gamma_1 - 4)n + (\beta_1 - 1)$$

We get $\gamma_0 = 0,\ \gamma_1 = 4,\ \beta_0 = 0,\ \beta_1 = 1$ and

Solution 3:

$$\alpha = 1,\ \ \gamma_0 = 0,\ \gamma_1 = 4,\ \beta_0 = 0,\ \beta_1 = 1$$

Equation 5

CF: $h(n) = \alpha A(n) + \gamma_0 B(n) + \gamma_1 C(n) + \beta_0 D(n) + \beta_1 E(n)$

We evaluate CF for **Solution 3:**

$\alpha = 1,\ \gamma_0 = 0,\ \gamma_1 = 4,\ \beta_0 = 0,\ \beta_1 = 1$ and get

CF = RF iff the following holds

Fact 5 For all $n \in N - \{0\}$

$$A(n) + 4C(n) + E(n) = n^2$$

This is our equation 5

Repertoire Method: System of Equations

We obtained the following system of **5 equations** on A(n), B(n), C(n), D(n), E(n)

1. $\alpha A(n) + \beta_0 D(n) + \beta_1 E(n) = (\alpha, \beta_{b_{m-1}}, ...\beta_{b_0})_4$

2. $A(n) = 4^k$

3. A(n) - 3D(n) - 3E(n) = 1

4. A(n) - 2B(n) - 2 C(n) + E(n) = n

5. $A(n) + 4C(n) + E(n) = n^2$

We solve it and put the solution into

$h(n) = \alpha A(n) + \gamma_0 B(n) + \gamma_1 C(n) + \beta_0 D(n) + \beta_1 E(n)$

System of Equations Solution

First solution formulas

$$
\begin{aligned}
A(n) &= 4^k \\
B(n) &= \frac{A(n)-3C(n)-1}{3} \quad \text{(from \textbf{3.})} \\
D(n) &= \frac{3A(n)+3C(n)-n^2-2n}{4} \\
E(n) &= \frac{n^2-A(n)-C(n)}{4} \quad \text{(from \textbf{5.})} \\
C(n) &= ((\alpha\beta_{b_{m-1}}...\beta_{b_1}\beta_{b_0})_4 - \alpha \cdot 4^k - \beta_0(1-4^k)/3)/(\beta_0+\beta_1)
\end{aligned}
$$

Problem 20 Solution

After substitution we obtain:

$$
\begin{aligned}
A(n) &= 4^k \\
B(n) &= \frac{4^k - 3C(n) - 1}{3} \\
D(n) &= \frac{3\ 4^k + 3C(n) - n^2 - 2n}{4} \\
E(n) &= \frac{n^2 - 4^k - C(n)}{4} \\
C(n) &= ((\alpha\beta_{b_{m-1}} \ldots \beta_{b_1}\beta_{b_0})_4 - \alpha \cdot 4^k - \beta_0(1 - 4^k)/3)/(\beta_0 + \beta_1)
\end{aligned}
$$

General Remark

Observe that the use of the **Relaxed k -Radix Representation** partial solution, i.e. the **equation 1** as well as proving the formula for A(n), i.e. the **equation 2** were essential for the **solution of the problem**

Like in Ch1 **Problem 16** - the other typical repertoire functions like $h(n) = n^3$ etc . **FAIL**, i.e. lead to contradictions - easy and left for you to evaluate!

Read carefully published Solutions to Ch1**Problem 16**, which is, in fact an easier case of **Problem 20** just presented in a FULL DETAIL

Observe also that we have **proved** all **Relaxed k -Radix Representation** formulas needed and you have to KNOW these proofs for your tests