# cse547, math547
# DISCRETE MATHEMATICS

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# LECTURE 4

# CHAPTER 1

## PART FIVE: Binary and Relaxed Binary Solutions for Generalized Josephus

## Binary Solution

We proved that the **original** J-recurrence:

J(1) = 1, J(2n) = 2J(n) - 1, J(2n+1) = 2J(n) + 1 for $n > 1$

has a beautiful **binary** CF **solution**

$$J((b_m, b_{m-1}, ...b_1, b_0)_2) = (b_{m-1}, b_{m-2}, ...b_0, b_m)_2,$$

move $b_m$ !

where $b_m = 1$, as $n = 2^m + l$

**Question:** Does the **generalized Josephus** *GJ* admits a similar solution?

**Answer:** YES.

## Generalized Josephus GF

We **generalized** the function J to function
$f: N - \{0\} \longrightarrow N$ defined as follows

$$f(1) = \alpha$$

$$f(2n) = 2f(n) + \beta, \quad n \geq 1$$

$$f(2n+1) = 2f(n) + \gamma, \quad n \geq 1$$

Observe that J = f for $\alpha = 1, \beta = -1, \gamma = 1$

We call the function f a **Generalized Josephus** GJ

# New Formula for GJ

We re-write the function f as follows

$$f(1) = \alpha;$$
$$f(2n + j) = 2f(n) + \beta_j$$
$$\text{for } j = 0, 1, \qquad n \geq 1$$

Assume

$$k = (b_m, b_{m-1}, ...b_1, b_0)_2$$

We want to evaluate:

$$f(k) = f((b_m, b_{m-1}, ...b_1, b_0)_2)$$

# Binary Representation for k=2n

Consider case when

$$k = 2n + 0, \quad j = 0.$$

The binary representation of $k = 2n$ is given as:

$$2n = (b_m, b_{m-1}, ...b_1, b_0)_2$$

$$2n = 2^m b_m + b_{m-1} + ... + 2b_1 + b_0$$

## Binary Representation for k=2n

We get $b_m = 1$ and $b_0 = 0$

**Hence,**

$$n = 2^{m-1} b_m + ... + b_1$$

$$\mathbf{n = (b_m, b_{m-1}, ...b_1)_2}$$

**Question**: What happens when $k = 2n + 1, \;\; j = 1$?

# Binary Representation for k=2n+1

Consider case when $k = 2n + j,\ j = 1$

The binary representation of k=2n +1 is given as:

$$2n + 1 = (b_m, b_{m-1}, ...b_1, b_0)_2$$

$$2n + 1 = 2^m b_m + b_{m-1} + ... + 2b_1 + b_0$$

$$b_0 = 1, b_m = 1$$

## Binary Representation for k=2n+1

We get

$$2n + 1 = 2^m b_m + b_{m-1} + ... + 2b_1 + 1$$

$$2n = 2^m b_m + b_{m-1} + ... + 2b_1$$

$$n = 2^{m-1} b_m + b_{m-1} + ... + b_1$$

$$\mathbf{n = (b_m, b_{m-1}, ... b_1)_2}$$

# Binary Representation

We have **proved** that whether we have a binary representation of $2n = (b_m, b_{m-1}, ...b_1, b_0)_2$

or a binary representation of 2n+1 = $(b_m, b_{m-1}, ...b_1, b_0)_2$,

the corresponding representations of n are the same:

$$\mathbf{n = (b_m, b_{m-1}, ...b_1)_2}$$

**Fact**

When dealing with **binary representation** we do not need to consider cases of $\mathbf{n \in odd}$ or $\mathbf{n \in even}$

when using our recursive formula

$$f(2n+j) = 2f(n) + \beta_j, \qquad j = 0, 1$$

## CF in Binary Representation

Here is our recursive formula

RF: $f(1) = \alpha, \quad f(2n) = 2f(n) + \beta_0, \quad f(2n+1) = 2f(n) + \beta_1$

By the **Fact** evaluate can write RF using $n$ in **binary representation**

$$f((b_m, b_{m-1}, ...b_1, b_0)_2) = 2f((b_m, b_{m-1}, ...b_1)_2) + \beta_{b_i},$$

where

$$\beta_{b_j} = \begin{cases} \beta_0 & b_j = 0 \\ \beta_1 & b_j = 1 \end{cases} \qquad j = 0...m-1$$

# CF in Binary Representation

We evaluate:

$$
\begin{aligned}
\mathbf{f((b_m, b_{m-1}, ...b_1, b_0)_2)} &= 2f((b_m, b_{m-1}, ..., b_1)_2) + \beta_{b_0} \\
&= 2(2f((b_m, b_{m-1}, ..., b_2)_2) + \beta_{b_1}) + \beta_{b_0} \\
&= 4f((b_m, b_{m-1}, ..., b_2)_2) + 2\beta_{b_1} + \beta_{b_0} \\
&\vdots \\
&= 2^m f((b_m)_2) + 2^{m-1}\beta_{b_{m-1}} + ... + 2\beta_{b_1} + \beta_{b_0} \\
&= \mathbf{2^m f((1)_2) + 2^{m-1}\beta_{b_{m-1}} + ... + 2\beta_{b_1} + \beta_{b_0}}
\end{aligned}
$$

## CF in Binary Representation

We know that $f(1) = \alpha$

So we get (almost) CF formula

$$\mathbf{f((b_m, b_{m-1}, ...b_1, b_0)_2) = 2^m}\alpha + \mathbf{2^{m-1}}\beta_{\mathbf{b_{m-1}}} + ... + \mathbf{2}\beta_{\mathbf{b_1}} + \beta_{\mathbf{b_0}}$$

where

$$\beta_{b_j} = \begin{cases} \beta_0 & b_j = 0 \\ \beta_1 & b_j = 1 \end{cases} \qquad j = 0...m-1$$

# Relaxed Binary CF

We **define** a relaxed binary representation as follows

$$\mathbf{2^m}\alpha + \mathbf{2^{m-1}}\beta_{\mathbf{b_{m-1}}} + ... + \beta_{\mathbf{b_0}} = (\alpha, \beta_{\mathbf{b_{m-1}}}, ..\beta_{\mathbf{b_0}})_\mathbf{2}$$

where now $\beta_{b_k}$ are now any numbers, not only 0,1

We write the relaxed binary CF as

$$\begin{aligned} \mathbf{f}((\mathbf{b_m}, \mathbf{b_{m-1}}, ...\mathbf{b_1}, \mathbf{b_0})_\mathbf{2}) &= (\alpha, \beta_{\mathbf{b_{m-1}}}, ..\beta_{\mathbf{b_0}})_\mathbf{2} \\ ''\textit{normal}'' &= \textit{relaxed} \end{aligned}$$

$$\beta_{b_j} = \begin{cases} \beta_0 & b_j = 0 \\ \beta_1 & b_j = 1 \end{cases} \qquad j = 0, ..., m-1$$

# Example: Original Josephus

The GJ function f becomes the **original Josephus** when $\beta_0 = -1, \beta_1 = 1$

**Example**

Let n = 100

Use the **relaxed binary** CF to show that f(100) = 73 =J(n) as we have already evaluated

$$n = (\ 1\ 1\ 0\ 0\ 1\ 0\ 0\ )_2$$

$$(b_6 b_5 b_4 b_3 b_2 b_1 b_0)$$

Relaxed coordinates are

$$\beta_{b_j} = \begin{cases} \beta_0 & b_j = 0 \\ \beta_1 & b_j = 1 \end{cases}$$

and hence

$$\beta_{b_j} = \begin{cases} -1 & b_j = 0 \\ 1 & b_j = 1 \end{cases}$$

## Example

We have

$$n = (\ 1\ 1\ 0\ 0\ 1\ 0\ 0\ )_2$$
$$(b_6 b_5 b_4 b_3 b_2 b_1 b_0)$$

$$\mathbf{f((b_m, b_{m-1}, ...b_1, b_0)_2)} = (\alpha, \beta_{\mathbf{b_{m-1}}}, ...\beta_{\mathbf{b_0}})_{\mathbf{2}}$$

*"normal"* = *relaxed*

$$\beta_{b_j} = \begin{cases} -1 & b_j = 0 \\ 1 & b_j = 1 \end{cases}$$

We evaluate

$$f(n) = f((\ 1\ 1\ 0\ 0\ 1\ 0\ 0\ )_2)) =^{relax} (\alpha,\ \beta_{b_5},\ \dots\ \beta_{b_0})$$

$$= (1, 1, -1, -1, 1, -1, -1)_2 = 64 + 32 - 16 - 8 + 4 - 2 - 1 = 73$$

## Cyclic - Shift Property

We **proved** that the original J-recurrence:

J(1) = 1, J(2n) = 2J(n) - 1, J(2n+1) = 2J(n) + 1 for $n > 1$

has a beautiful binary CF solution, called **cyclic - shift property**, namely

$$J((b_m, b_{m-1}, ...b_1, b_0)_2) = (b_{m-1}, b_{m-2}, ...b_0, b_m)_2$$

We prove now that the **cyclic - shift property** holds also for the GF function $f$ in the case when $\beta_0 = -1, \beta_1 = 1$, i.e.

$$f((b_m, b_{m-1}, ...b_1, b_0)_2) = (b_{m-1}, b_{m-2}, ...b_0, b_m)_2$$

We know that $b_m = 1$, so we have to prove that:

$$f(1, b_{m-1}, ...b_1, b_0)_2) = (b_{m-1}, b_{m-2}, ...b_0, 1)_2,$$

for $f$ such that $\beta_0 = -1, \beta_1 = 1$

## Cyclic - Shift Property for GJ

We have proved the **relaxed binary** CF solution for GJ:

$$CF: \quad f((1, b_{m-1}, ...b_1, b_0)_2) = (1, \beta_{b_{m-1}}, ...\beta_{b_0})_2$$

where $f(n)$ contains now 1 and -1 as defined by

$$\beta_{b_j} = \begin{cases} -1 & b_j = 0 \\ 1 & b_j = 1 \end{cases}$$

## Example

**EXAMPLE**

Consider $n = (1, 0, 0, 1, 0, 0, 1)_2$

By CF we have that

$$f((1, 0, 0, 1, 0, 0, 1)_2) = (1, -1, -1, 1, -1, -1, 1)_2$$

**General Observation**

$f$ transforms a BLOCK of 0's in **normal binary representation** into a BLOCK of -1's in the **relaxed representation**

$$\mathbf{f((1, 0, 0 ... 0)_2) = (1, -1, -1 ... -1)_2}$$

## ONE BLOCK Transformation

We **prove** now the following relationship between relaxed and **normal** representation

ONE BLOCK **transformation**

$$(\mathbf{1}, -\mathbf{1}, -\mathbf{1} \ldots, -\mathbf{1})_{\mathbf{2}} = (\mathbf{0}, \mathbf{0}, \mathbf{0} \ldots, \mathbf{0}, \mathbf{1})_{\mathbf{2}}$$

**Proof**: Let $n = ((-1, -1 \ldots, -1)_2$

$$
\begin{aligned}
\mathbf{n} = (\mathbf{1}, -\mathbf{1}, -\mathbf{1} \ldots, -\mathbf{1})_{\mathbf{2}} \quad &=^{def} \quad 2^m - 2^{m-1} - 2^{m-2} - \ldots - 2^1 - 2^0 \\
&= \quad 2^{m-1} - 2^{m-2} - \ldots - 2^1 - 2^0 \\
&= \quad 2^{m-2} - 2^{m-3} - \ldots - 2^1 - 2^0 \\
& \quad\quad \vdots \\
&= \quad 2^1 - 2^0 \\
&= \quad 2 - 1 \\
&= \quad \mathbf{1} = (\mathbf{0}, \mathbf{0}, \mathbf{0}, \mathbf{0}, \mathbf{1})_{\mathbf{2}}
\end{aligned}
$$

# Many Blocks Transformation

**Example** for TWO BLOCKS **transformation** plus **binary shift**

$$
\begin{aligned}
f((1,0,0,1,1,0,0,1)_2) &= (1,-1,-1,1,1,-1,-1,1)_2 \\
&=^{1bt} (0,0,1,1,1,-1,-1,1)_2 \\
&=^{1bt} (0,0,1,1,0,0,1,1)_2 \\
&= (0,0,1,1,0,0,1,1)_2
\end{aligned}
$$

We know that $f((b_m, ..., b_1, b_0)_2) = (\alpha, \beta_{b_{m-1}}, ..., \beta_{b_0})_2$ **OBSERVE** that each block of binary digits $(1, 0..0)_2$ is **transformed** by $f$ into $(1, -1, ...)_2$ and multiple applications of **one block transformation** transforms them **back** to $(1, 0..0)_2$, so

$$((\alpha, \beta_{\mathbf{b_{m-1}}}, \ldots, \beta_{\mathbf{b_0}})_\mathbf{2} =^{\mathbf{mbt}} (\mathbf{b_{m-1}, ...b_1, b_0, 1})_\mathbf{2}$$

where mbt denotes multiple BLOK transformations, and we know that $\alpha = 1$

## Cyclic - Shift Property

We now evaluate:

$$\begin{aligned} f((1, b_{m-1}, ..., b_1, b_0)_2) &= (\alpha, \beta_{b_{m-1}}, ..., \beta_{b_0})_2 \\ &=^{mbt} (b_{m-1}, ..., b_1, b_0, 1)_2 \end{aligned}$$

This ends the proof of the Cyclic - Shift Property for Generalized Josephus f with $\alpha = 1,\ \beta_0 = -1,\ \beta_1 = 1$

## Exercise 1

Given

$$\begin{aligned} f(1) &= 5 \\ f(2n) &= 2f(n) - 10 \\ f(2n+1) &= 2f(n) + 83 \end{aligned}$$

**Exercise 1**

Evaluate f(100)

**Solution**: just apply proper formulas!

# Exercise 2

Given

$$\begin{aligned} f(1) &= 5 \\ f(2n) &= 3f(n) - 10 \\ f(2n+1) &= 3f(n) + 83 \end{aligned}$$

**Exercise 2**

Evaluate f(100)

**Observe** that now we don't have proper formulas! They work only for base 2!

Goal **Generalize** f and develop **new formulas** (if possible)

# RADIX Representation

We **proved** while solving the Generalized Josephus that

RF: $f(1) = \alpha, \quad f(2n + j) = 2f(n) + \beta_j$

where $j = 0, 1$ and $n \geq 0$

has a **relaxed binary** CF formula

$$CF: \quad \mathbf{f((1, b_{m-1}, ...b_1, b_0)_2)} = (\alpha, \beta_{\mathbf{b_{m-1}}}, ...\beta_{\mathbf{b_0}})_{\mathbf{2}}$$

where $\beta_{b_j}$ are defined by

$$\beta_{b_j} = \begin{cases} \beta_0 & b_j = 0 \\ \beta_1 & b_j = 1 \end{cases} \qquad j = 0, ..., m-1$$

and where the **relaxed binary representation** is defined as

$$(\alpha, \beta_{\mathbf{b_{m-1}}}, ..., \beta_{\mathbf{b_0}})_{\mathbf{2}} = \mathbf{2^m}\alpha + \mathbf{2^{m-1}}\beta_{\mathbf{m-1}} + ... + \beta_{\mathbf{b_0}}$$

# Relaxed Radix Representation

We generalize GJ as follows

RF: $f(1) = \alpha, f(2n + j) = kf(n) + \beta_j,$

where $k \geq 2$, $j = 0, 1$ and $n \geq 0$

**Exercise:** PROVE that RF has a **relaxed** k-**representation** closed formula

$$CF: \quad \mathbf{f((1, b_{m-1}, ...b_1, b_0)_2)} = (\alpha, \beta_{\mathbf{b_{m-1}}}, ...\beta_{\mathbf{b_0}})_{\mathbf{k}}$$

where $\beta_{b_j}$ are defined as before by

$$\beta_{b_j} = \begin{cases} \beta_0 & b_j = 0 \\ \beta_1 & b_j = 1 \end{cases} ; \quad j = 0, ..., m - 1$$

and where we define the **relaxed** k- **representation** as follows

## Relaxed k-Radix Representation

**Definition**

**A relaxed** k- **representation** is defined as

$$(\alpha, \beta_{b_{m-1}}, ..., \beta_{b_0})_k = \alpha k^m + k^{m-1}\beta_{m-1} + ... + \beta_{b_0}$$

We repeat the **proof i** directly from there definition following the proof for the case $k = 2$

# Proof

$$
\begin{aligned}
f((b_m, b_{m-1}, ..., b_1, b_0)_2) &= k f((b_m, b_{m-1}, ..., b_1)_2) + \beta_{b_0} \\
&= k(k f((b_m, b_{m-1}, ..., b_2)_2) + \beta_{b_1}) + \beta_{b_0} \\
&= k^2 f((b_m, b_{m-1}, ..., b_2)_2) + k\beta_{b_1} + \beta_{b_0} \\
&= k^3 f((b_m, b_{m-1}, ..., b_3)_2) + k^2\beta_{b_2} + k\beta_{b_1} + \beta_{b_0} \\
&\vdots \\
&= k^m f((b_m)_2) + k^{m-1}\beta_{b_{m-1}} + ... + k\beta_{b_1} + \beta_{b_0} \\
&= k^m \alpha + k^{m-1}\beta_{b_{m-1}} + ... + k^2\beta_{b_2} + k\beta_{b_1} + \beta_{b_0} \\
&= (\alpha, \beta_{b_{m-1}}, ..., \beta_{b_1}, \beta_{b_0})_k
\end{aligned}
$$

$$
\begin{aligned}
f((b_m, b_{m-1}, ..., b_1, b_0)_2) &= (\alpha, \beta_{b_{m-1}} ... \beta_{b_1}, \beta_{b_0})_k \\
\text{base 2} &\rightarrow \text{base k}
\end{aligned}
$$

# Example

$$
\begin{aligned}
f(1) &= 5 \\
f(2n) &= 6f(n) + 3 \\
f(2n+1) &= 6f(n) - 10
\end{aligned}
$$

$$\alpha = 5$$

$$\beta_0 = 3$$

$$\beta_1 = -10$$

**Evaluate**: $f(100)$

# Example Solution

$$\alpha = 5,\ \beta_0 = 3,\ \beta_1 = -10,\ k = 6,\ n \;=\; (\ 1\ 1\ 0\ 0\ 1\ 0\ 0\ )_2$$

$$(b_6 b_5 b_4 b_3 b_2 b_1 b_0)$$

$$\beta_{b_j} = \begin{cases} \beta_0 & b_j = 0 \\ \beta_1 & b_j = 1 \end{cases}, \qquad j = 0, ..., m-1,$$

$$\beta_{b_0} = 3,\ \beta_{b_1} = 3,\ \beta_{b_2} = -10,\ \beta_{b_3} = 3;\ \beta_{b_4} = 3,$$

$$\beta_{b_5} = -10,\ \beta_{b_6} = -10$$

$$\mathbf{f(100) = f((\ 1\ 1\ 0\ 0\ 1\ 0\ 0\ )_2) = (5, -10, 3, 3, -10, 3, 3)_6}$$

# More General Function

Given

RF:

$$f(i) = \alpha_i, \qquad i = 1, ..., d-1$$

$$f(dn+j) = cf(n) + \beta_j, \qquad n \geq 1, \ \ 0 \leq j < d$$

**Exercise**

**Prove** the following closed formula CF:

$$\mathbf{f((b_m, b_{m-1}, ..., b_1, b_0)_d) \ = \ (\alpha_{b_m}, \beta_{b_{m-1}} ... \beta_{b_1}, \beta_{b_0})_c}$$

# Example

$$
\begin{aligned}
f(1) &= 34 \\
f(2) &= 5 \\
f(3n) &= 10f(n) + 76 \\
f(3n+1) &= 10f(n) - 2 \\
f(3n+2) &= 10f(n) + 8
\end{aligned}
$$

$$
\beta_{b_j} = \begin{cases} \beta_0 & b_j = 0 \\ \beta_1 & b_j = 1 \\ \beta_2 & b_j = 2 \end{cases}, \qquad j = 0, ..., d-1,
$$

## Example Solution

We evaluate:

$$i = 1, 2$$
$$j = 0, 1, 2$$

$$d = 3$$
$$c = 10$$

$$\alpha_1 = 34$$
$$\alpha_2 = 5$$
$$\beta_0 = 76$$
$$\beta_1 = -2$$
$$\beta_2 = 8$$

# Example

**Evaluate:** $f(19)$

$$19 = (201)_3 = 2 \cdot 3^2 + 0 \cdot 3 + 1$$

$$\alpha_{b_2} = \alpha_2 = 5$$
$$\beta_{b_0} = \beta_0 = 76$$
$$\beta_{b_1} = \beta_1 = -2$$

$$\begin{aligned} f(19) &= f((201)_3) \\ &= (5, 76, -2)_{10} \\ &= 5 \cdot 10^2 + 76 \cdot 10 - 2 \\ &= 500 + 760 - 2 \\ &= 1258 \end{aligned}$$

## Short Solution

$$\mathbf{f}((\mathbf{b_m}, \mathbf{b_{m-1}}, ..., \mathbf{b_1}, \mathbf{b_0})_{\mathbf{d}}) \quad = \quad (\alpha_{\mathbf{b_m}}, \beta_{\mathbf{b_{m-1}}} .. \beta_{\mathbf{b_1}}, \beta_{\mathbf{b_0}})_{\mathbf{c}}$$

Take

$$19 = (2\ 0\ 1)_3$$

**Corresponding solution** is

$$(\alpha_2,\ \ \beta_0,\ \ \beta_1)_{10}$$

we evaluate $\alpha_2 = 5, \ \ \beta_0 = 76, \ \ \beta_1 = -2$ and get

**Solution:**

$$(5,\ \ 76,\ \ -2)_{10}$$