cse547, math547
DISCRETE MATHEMATICS

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LECTURE 11a

CHAPTER 3
INTEGER FUNCTIONS

PART1: Floors and Ceilings

PART 2: Floors and Ceilings Applications

PART 2
Floors and Ceilings Applications

Casino Problem

Reminder of Casino Problem

There is a roulette wheel with 1,000 slots numbered $1 \ldots 1{,}000$

IF the number n that comes up on a spin is divisible by $\lfloor \sqrt[3]{n} \rfloor$, i.e. $\sqrt[3]{n}\rfloor \mid n$

THEN n is the **winner**

The summations becomes

$$W = \sum_{n=1}^{1000} [\text{n is a winner}] = \sum_{n=1}^{1000} [\lfloor \sqrt[3]{n} \rfloor \mid n]$$

where we **define divisibility** $|$ in a standard way

$k \mid n$ if and only if there exists $m \in Z$ such that $n = km$

Book Solution

Here are 7 steps of our BOOK solution

1 $W = \sum_{n=1}^{1000} [n \text{ is a winner}] = \sum_{n=1}^{1000} \left[\lfloor \sqrt[3]{n} \rfloor \mid n\right]$

2 $W = \sum_{k,n} \left[k = \lfloor \sqrt[3]{n} \rfloor\right] [k|n] [1 \leq n \leq 1000]$

3 $W = \sum_{k,n,m} \left[k^3 \leq n < (k+1)^3\right] [n = km] [1 \leq n \leq 1000]$

4 $W = 1 + \sum_{k,m} \left[k^3 \leq km < (k+1)^3\right] [1 \leq k < 10]$

5 $W = 1 + \sum_{k,m} \left[m \in \left[k^2 \ldots \frac{(k+1)^3}{k}\right)\right] [1 \leq k < 10]$

6 $W = 1 + \sum_{1 \leq k < 10} \left(\lceil k^2 + 3k + 3 + \frac{1}{k} \rceil - \lceil k^2 \rceil\right)$

7 $W = 1 + \sum_{1 \leq k < 10} (3k+4) = 1 + \frac{7+31}{2} 9 = 172$

Class Problem

Here are the BOOK comments

1. This derivation merits careful study

2. The only "difficult" maneuver is the decision between lines **3** and **4** to treat n =1000 as a special case

3. The inequality $k^3 \leq n < (k+1)^3$ does not combine easily with $1 \leq n \leq 1000$ when k=10

Book Solution Comments

Class Problem

Write down explanation of **each step** with **detailed** justifications (Facts, definitions) why they are correct

By doing so fill all gaps in the **proof** that

$$W = \sum_{n=1}^{1000} [\lfloor \sqrt[3]{n} \rfloor \mid n] = 172$$

This problem can also appear on your tests

QUESTIONS about Book Solution

Here are **questions** to answer about the steps in the BOOK solution

1 W = $\sum_{n=1}^{1000}[n \textit{ is a winner }] = \sum_{n=1}^{1000}\left[\lfloor \sqrt[3]{n} \rfloor \mid n\right]$

Q1 Explain why $[n \textit{ is a winner }] = \left[\lfloor \sqrt[3]{n} \rfloor \mid n\right]$

2 W = $\sum_{k,n}\left[k = \lfloor \sqrt[3]{n} \rfloor\right][k|n][1 \leq n \leq 1000]$

Q2 Explain why and how we have changed a sum $\sum_{n=1}^{1000}$ into a sum $\sum_{k,n}$ and

$$\sum_{n=1}^{1000}\left[\lfloor \sqrt[3]{n} \rfloor \mid n\right] = \sum_{k,n}\left[k = \lfloor \sqrt[3]{n} \rfloor\right][k|n][1 \leq n \leq 1000]$$

QUESTIONS about Book Solution

3 $W = \sum_{k,n,m} \left[k^3 \le n < (k+1)^3 \right] [n = km] [1 \le n \le 1000]$

Q3 Explain why

$$\left[k = \lfloor \sqrt[3]{n} \rfloor \right] [k|n] = \left[k^3 \le n < (k+1)^3 \right] [n = km]$$

Explain why and how we have changed sum $\sum_{k,n}$ into a sum $\sum_{k,n,m}$

QUESTIONS about Book Solution

4 W = 1 + $\sum_{k,m} \left[k^3 \leq km < (k+1)^3 \right] [1 \leq k < 10]$

Q4 There are three sub- questions; the last one is one of the book questions

1. Explain why

$$\left[k^3 \leq n < (k+1)^3 \right] [n = km] [1 \leq n \leq 1000] =$$
$$\left[k^3 \leq km < (k+1)^3 \right] [1 \leq k < 10]$$

2. Explain why and how we have changed sum $\sum_{k,n,m}$ into
a sum $\sum_{k,m}$

3. Explain HOW and why we have got $1 + \sum_{k,m}$

QUESTIONS about Book Solution

5 $$W = 1 + \sum_{k,m} \left[m \in \left[k^2 \ldots \frac{(k+1)^3}{k} \right) \right] [1 \le k < 10]$$

Q5 Explain transition

$$\left[k^3 \le km < (k+1)^3 \right] = \left[m \in \left[k^2 \ldots \tfrac{(k+1)^3}{k} \right) \right]$$

QUESTIONS about Book Solution

6 W = 1 + $\sum_{1 \le k < 10} \left(\lceil k^2 + 3k + 3 + \frac{1}{k} \rceil - \lceil k^2 \rceil \right)$

Q6 Explain (prove) why

$$\sum_{k,m} \left[m \in \left[k^2 \ldots \frac{(k+1)^3}{k} \right) \right] [1 \le k < 10] =$$

$$\sum_{1 \le k < 10} \left(\lceil k^2 + 3k + 3 + \frac{1}{k} \rceil - \lceil k^2 \rceil \right)$$

Observe that $\left[m \in \left[k^2 \ldots \frac{(k+1)^3}{k} \right) \right]$ is a characteristic function and $\left(\lceil k^2 + 3k + 3 + \frac{1}{k} \rceil - \lceil k^2 \rceil \right)$ is an integer

QUESTIONS about Book Solution

7 W = 1 + $\sum_{1 \le k < 10} (3k+4) = 1 + \frac{7+31}{2} 9 = 172$

Q7 Explain (prove) why

$$\left(\lceil k^2 + 3k + 3 + \frac{1}{k} \rceil - \lceil k^2 \rceil\right) = (3k+4)$$

Before we giving answers to Q1 - Q7 we need to review some of the SUMS material

SUMS - a Short Review

Definition 1

$$\sum_{P(k)} a_k = \sum_{k \in K} a_k = \sum_{k} [P(k)]\, a_k = \sum_{k} [k \in K]\, a_k$$

where $K = \{ k \in N : P(k) \}$ and K is FINITE

and $[P(k)]$ is a characteristic function of P(k)

$$[P(k)] = \begin{cases} 1 & P(k) \ \ true \\ 0 & P(k) \ \ false \end{cases}$$

Property 1

Let's take a particular case when the sequence $a_k = 1$ for all $k \in N$

Directly from the **Definition 1** we get the following

Property 1

$$\sum_k [P(k)] = \sum_{k \in K} 1 = |K|$$

where $|K|$ denotes the number of elements of the set K

We re-write is also as

$$\sum_k [P(k)] = \sum_{P(k)} 1 = |P(k)|$$

Definition 2

Definition 2

In a case of multiple sums (here a double sum) we define

$$\sum_{k\in K,m\in M} a_{k,m} = \sum_{P(k),Q(m)} a_{k,m} = \sum_{Q(m)}\sum_{P(k)} a_{k,m} = \sum_{P(k)}\sum_{Q(m)} a_{k,m}$$

and

$$\sum_{P(k),Q(m)} a_{k,m} = \sum_{k,m} a_{k,m}[P(k)][Q(m)]$$

where

$K = \{k \in N : P(k)\ \}$ and $M = \{m \in N : Q(m)\ \}$

Triple and many-multiple sums definitions are similar

Property 2

Let's take a particular case when the sequence

$a_{k,m} = 1$ for all $k, m \in N$

Directly from the **Definition 2** and **Property 1** we get the following

Property 2

$$\sum_{k,m} [P(m)]\,[Q(k)] = \sum_{Q(k)} \sum_{P(m)} 1 = \sum_{Q(k)} |\, P(m) \,|$$

where we denote for short

$$|\, P(m) \,| = |\, \{ m \in N : P(m) \} \,|$$

Characteristic Functions

We have proved the following properties of characteristic functions

F1 For any predicates P(k), Q(k)

$$[P(k) \cap Q(k)] = [P(k)][Q(k)]$$

F2 For any predicates P(k), Q(k)

$$[P(k) \cup Q(k)] = [P(k)] + [Q(k)] - [P(k) \cap Q(k)]$$

Property 3

From **Property 1** and **F2** we get directly the following **Property 3**

$$\sum_k [P(k) \cup Q(k)] = \sum_k [P(k)] + \sum_k [Q(k)] - \sum_k [P(k) \cap Q(k)]$$

where

$k \in K$ and $K = K_1 \times K_2 \cdots \times K_i$ for $1 \leq i \leq n$

Observe that the above formula represents single (i =1) or multiple $(i > 1)$ sums

It is a particular case of the Combined Domains Property (next slide) - just a reminder!

Combined Domains Property

Here is the Combined Domains Property

Property 4

$$\sum_{Q(k) \cup R(k)} a_k = \sum_{Q(k)} a_k + \sum_{R(k)} a_k - \sum_{Q(k) \cap R(k)} a_k$$

where, as before,

$k \in K$ and $K = K_1 \times K_2 \cdots \times K_i$ for $1 \leq i \leq n$

and the above formula represents single (i =1) and multiple ($i > 1$) sums

Book Solution Step 1

Here are the answers to the questions about the steps in the BOOK solution

1 $$W = \sum_{n=1}^{1000} [n \text{ is a winner}] = \sum_{n=1}^{1000} [\lfloor \sqrt[3]{n} \rfloor \mid n]$$

Answer 1

Definition of the winner in the Casino Problem

Book Solution Step 2

2 W = $\sum_{k,n} [k = \lfloor \sqrt[3]{n} \rfloor] \, [k|n] \, [1 \leq n \leq 1000]$

Answer 2 Take $P(n) \equiv \lfloor \sqrt[3]{n} \rfloor \mid n$

We transform $P(n)$ introducing a new variable k

$$P(n) \equiv \lfloor \sqrt[3]{n} \rfloor \mid n \equiv (k = \lfloor \sqrt[3]{n} \rfloor) \cap (k \mid n)$$

We use it to transform the one variable sum to a two variable sum as follows

$$\sum_{n=1}^{1000} [\lfloor \sqrt[3]{n} \rfloor \mid n] = \sum_{k,n} [(k = \lfloor \sqrt[3]{n} \rfloor) \cap (k \mid n)] \, [1 \leq n \leq 1000]$$

Hence we get

Book Solution Step 2

We use the property **F1** of Characteristic Functions

$$[P(k) \cap Q(k)] \; = \; [P(k)][Q(k)]$$

and we get **2.**

$$\sum_{n=1}^{1000} \left[\lfloor \sqrt[3]{n} \rfloor \mid n \right] = \sum_{k,n} [(k = \lfloor \sqrt[3]{n} \rfloor)] \; [(k \mid n)] \; [1 \leq n \leq 1000]$$

Book Solution Step 2

We use the definition of divisibility to further transform $P(n,k) \equiv (k = \lfloor \sqrt[3]{n} \rfloor) \cap (k \mid n)$ and and introduce another variable m

$$P(n,k) \equiv (\lfloor \sqrt[3]{n} \rfloor) \cap (k \mid n) \equiv (k = \lfloor \sqrt[3]{n} \rfloor) \cap (n = km)$$

We use it and the property **F1** of Characteristic Functions to transform the two variable sum **2** to a three variable sum

$$\sum_{k,n} [k = \lfloor \sqrt[3]{n} \rfloor] \, [k \mid n] \, [1 \leq n \leq 1000] =$$

$$= \sum_{k,n,m} [k = \lfloor \sqrt[3]{n} \rfloor] \, [n = km] \, [1 \leq n \leq 1000]$$

Book Solution Step 3

3 $W = \sum_{k,n,m} \left[k^3 \leq n < (k+1)^3\right] [n = km] [1 \leq n \leq 1000]$

Answer 3

We have already transformed **2** to a three variable sum

$$\sum_{k,n,m} [k = \lfloor \sqrt[3]{n} \rfloor] \ [n = km] \ [1 \leq n \leq 1000]$$

Now we use the property **8.**

$\lfloor x \rfloor = n$ if and only if $n \leq x < n+1$ to $k = \lfloor \sqrt[3]{n} \rfloor$ and we get

$$\lfloor \sqrt[3]{n} \rfloor = k \quad \text{if and only if} \quad k \leq \sqrt[3]{n} < k+1$$

and also

$$k \leq \sqrt[3]{n} < k+1 \quad \text{if and only if} \quad k^3 \leq n < (k+1)^3$$

Book Solution Step 3

We replace $k = \lfloor \sqrt[3]{n} \rfloor$ by $k^3 \leq n < (k+1)^3$ in already transformed **2**

$$\sum_{k,n,m} [k = \lfloor \sqrt[3]{n} \rfloor] \, [n = km] \, [1 \leq n \leq 1000]$$

and obtain

$$\sum_{k,n,m} [k^3 \leq n < (k+1)^3] \, [n = km] \, [1 \leq n \leq 1000]$$

and so we proved **3**

Book Solution Step 4

4 $W = 1 + \sum_{k,m} \left[k^3 \leq km < (k+1)^3\right] [1 \leq k < 10]$

Answer 4

We have proved that

$$W = \sum_{k,n,m} [k^3 \leq n < (k+1)^3]\, [n = km]\, [1 \leq n \leq 1000]$$

We want now to transform limits of the sum to contain only k, m, i.e. we want to eliminate n

Book Solution Step 4

Let's analyze the sum predicate

$$P \equiv (k^3 \leq n < (k+1)^3) \cap (n = km) \cap (1 \leq n \leq 1000)$$

Observe that when $(k+1)^3 = 1000,\ k+1 = 10, k = 9$
and $1 \leq k < 10$

We almost eliminated n - we miss $n = 1000$

It means we get

$$P \equiv ((k^3 \leq n < (k+1)^3) \cap (n = km) \cap (1 \leq k < 10)\ ) \cup (n = 1000)$$

and hence

$$[k^3 \leq n < (k+1)^3]\ [n = km]\ [1 \leq n \leq 1000]$$

$$= [((k^3 \leq km < (k+1)^3) \cap (1 \leq k < 10)\ ) \cup (km = 1000)]$$

Book Solution Step 4

So now we get

$$W = \sum_{k,m} [((k^3 \leq km < (k+1)^3) \cap (1 \leq k < 10)\ ) \cup (km = 1000)]$$

We use now the **Property 3**

$$\sum_{k,m} [P \cup Q] = \sum_{k,m} [P] + \sum_{k,m} [Q] - \sum_{k,m} [P \cap Q]$$

for $P \equiv ((k^3 \leq km < (k+1)^3) \cap (1 \leq k < 10)\ )$ and $Q \equiv (km = 1000)$

Book Solution Step 4

Denote $P \equiv ((k^3 \leq km < (k+1)^3) \cap (1 \leq k < 10)\ )$ and $Q \equiv (km = 1000)$

We get

$$W = \sum_{k,m} [P] + \sum_{k,m} [km = 1000] - \sum_{k,m} [P \cap Q]$$

where

$$\sum_{k,m} [P] = \sum_{k,m} [(k^3 \leq km < (k+1)^3)]\ [1 \leq k < 10]$$

The **Property 1** says

$$\sum_{k} [P(k)] = \sum_{P(k)} 1 = \mid P(k) \mid$$

so we get that

$$\sum_{k,m} [km = 1000] = \mid \{n : \ n = km = 1000\} \mid = \mid \{n : \ n = 1000\} \mid = 1$$

Book Solution Step 4

We proved that

$$W = 1 + \sum_{k,m} [P] + \; - \sum_{k,m} [P \cap Q]$$

Now we have to evaluate $P \cap Q$

$$P \cap Q \equiv ((k^3 \leq km < (k+1)^3) \cap (1 \leq k < 10)\;) \cap (km = 1000)$$

$$P \cap Q \equiv (k^3 \leq 1000 < (k+1)^3) \cap (1 \leq k \leq 9\;)$$

$$CONTRADICTION: \quad 9^3 \leq 1000 < 10^3$$

This means that $\sum_{k,m} [P \cap Q] = 0$ and

$$W = 1 + \sum_{k,m} [k^3 \leq km < (k+1)^3]\,[1 \leq k < 10]$$

what ends the proof of **4**

Book Solution Step 5

Consider the Step 5

$$\textbf{5} \quad W = 1 + \sum_{k,m} \left[m \in \left[k^2 \ldots \frac{(k+1)^3}{k} \right) \right] [1 \leq k < 10]$$

Answer 5

Missing steps are as follows

First let's look again at the Step 4

$$W = 1 + \sum_{k,m} [k^3 \leq km < (k+1)^3] \, [1 \leq k < 10]$$

Dividing all sides of the inequality $k^3 \leq km < (k+1)^3$ by $k \geq 1$ we get

$$k^3 \leq km < (k+1)^3 \quad \text{iff} \quad k^2 \leq m < \frac{(k+1)^3}{k}$$

and by the definition of the interval

$$k^2 \leq m < \frac{(k+1)^3}{k} \quad \text{iff} \quad m \in [k^2 \ldots \frac{(k+1)^3}{k} \,)$$

Book Solution Step 5

We have proved that

$$k^3 \leq km < (k+1)^3 \quad \text{iff} \quad m \in \left[k^2 \ldots \frac{(k+1)^3}{k}\right)$$

and hence proved the transformation of the Step 4 into the Step 5 i.e. we proved

5 $W = 1 + \sum_{k,m} \left[m \in \left[k^2 \ldots \frac{(k+1)^3}{k}\right)\right] [1 \leq k < 10]$

Book Solution Step 6

Consider now

6 $W = 1 + \sum_{1 \le k < 10} \left(\lceil k^2 + 3k + 3 + \frac{1}{k} \rceil - \lceil k^2 \rceil \right)$

Let's now write all steps of transformation of the Step 5 into the Step 6

Observe that the transformation consists of proving that

$$\sum_{k,m} [m \in [k^2 \ldots \frac{(k+1)^3}{k})\ ]\ [1 \le k < 10] =$$

$$\sum_{1 \le k < 10} (\lceil k^2 + 3k + 3 + \frac{1}{k} \rceil - \lceil k^2 \rceil)$$

Book Solution Step 6

Consider the sum

$$\sum_{k,m} [m \in [k^2 \ldots \frac{(k+1)^3}{k})] \; [1 \le k < 10]$$

We apply the **Property 2**

$$\sum_{k,m} [P(m)][Q(k)] = \sum_{Q(k)} \sum_{P(m)} 1 = \sum_{Q(k)} \mid P(m) \mid$$

to it for $Q(k) \equiv 1 \le k < 10$ and

$P(m) \equiv \; m \in [k^2 \ldots \frac{(k+1)^3}{k})$

Book Solution Step 6

Observe that $\mid P(m) \mid$ = number of integers in the interval $[k^2 \dots \frac{(k+1)^3}{k})$ and so by the the fact that interval $[\alpha \dots \beta)$ has $\lceil \beta \rceil - \lceil \alpha \rceil$ elements we get

$$\mid P(m) \mid = \left\lceil \frac{(k+1)^3}{k} \right\rceil - \left\lceil k^2 \right\rceil = \left\lceil k^2 + 3k + 3 + \frac{1}{k} \right\rceil - \left\lceil k^2 \right\rceil$$

and the sum

$$\sum_{Q(k)} \mid P(m) \mid = \sum_{1 \leq k < 10} \left(\left\lceil k^2 + 3k + 3 + \frac{1}{k} \right\rceil - \left\lceil k^2 \right\rceil \right)$$

This ends the transformation of Step 5 into Step 6 - and hence the proof of correctness (other then the fact it is printed in the BOOK!) of the Step 6

Book Solution Step 7

This is Step 7

7 W = 1 + $\sum_{1\leq k<10}(3k+4) = 1 + \frac{7+31}{2}9 = 172$

Pretty obvious step but still need to pay attention to a small detail!

We need to bring back property

$$\textbf{12.}\quad \lfloor x+n \rfloor = \lfloor x \rfloor + n \ \text{ and } \ \lceil x+n \rceil = \lceil x \rceil + n$$

to evaluate, as $k \geq 1$

$$\left\lceil k^2+3k+3+\frac{1}{k}\right\rceil - \left\lceil k^2 \right\rceil = k^2+3k+3+\left\lceil \frac{1}{k}\right\rceil - k^2 = 3k+4$$

Casino Problem Revisisted

Observe that the Casino Problem is just a dressed - up version of the following mathematical question :

Question

How many integers n, where $1 \leq n \leq 1000$, satisfy the property $\lfloor \sqrt[3]{n} \rfloor | n$?

Genaralized Question

How many integers n, where $1 \leq n \leq k$, satisfy the property $\lfloor \sqrt[3]{n} \rfloor | n$? for k any natural number and $k \geq 1000$

Homework Problem: write a detailed solution to the **Genaralized Question**

Spectrum Partitions

Spectrum

Definition

For any $\alpha \in R$ we define a SPECTRUM of α as

$$Spec(\alpha) = \{\lfloor \alpha \rfloor, \lfloor 2\alpha \rfloor, \lfloor 3\alpha \rfloor \cdots\}$$

Remark

For some $\alpha \in R$, the spectrum $Spec(\alpha)$ is a **multiset** i.e, it can contain repeating elements.

Examples

Let's look at some examples, to see how it works.

Spectrum Examples

Example 1 $\alpha = \frac{1}{2}$

$$\lfloor \alpha \rfloor = 0, \quad \lfloor 2\alpha \rfloor = 1, \quad \lfloor 3\alpha \rfloor = \lfloor \frac{3}{2} \rfloor = 1, \quad \lfloor 4\alpha \rfloor = \lfloor \frac{4}{2} \rfloor = 2, \cdots$$

$$Spec(\alpha) = Spec(\frac{1}{2}) = \{0, 1, 1, 2, 2, 3, 3, 4, 4, 5, \cdots\}$$

Observe that $Spec(\frac{1}{2})$ is a **multi set**

Spectrum Examples

Example 2 $\alpha = \sqrt{2}$

$$\lfloor \alpha \rfloor = \lfloor \sqrt{2} \rfloor = 1, \quad \lfloor 2\alpha \rfloor = \lfloor 2\sqrt{2} \rfloor = \lfloor 2.8 \rfloor = 2$$

$$\lfloor 3\alpha \rfloor = \lfloor 3\sqrt{2} \rfloor = \lfloor 4.2 \rfloor = 4, \qquad \lfloor 4\alpha \rfloor = \lfloor 5.6 \rfloor = 5 \cdots$$

$$Spec(\sqrt{2}) = \{\lfloor \sqrt{2} \rfloor, \lfloor 2\sqrt{2} \rfloor, \ \lfloor 3\sqrt{2} \rfloor, \ldots\}$$

$$Spec(\sqrt{2}) = \{1, 2, 4, 5, 7, 8, 9, 11, 12, 14, 15, 16, \cdots\}$$

$$Spec(2 + \sqrt{2}) = \{\lfloor 2 + \sqrt{2} \rfloor, \lfloor 2(2 + \sqrt{2}) \rfloor, \ \lfloor 3(2 + \sqrt{2}) \rfloor, \ldots\}$$

$$Spec(2 + \sqrt{2}) = \{\lfloor 2 + \sqrt{2} \rfloor, \lfloor 4 + 2\sqrt{2} \rfloor, \ \lfloor 6 + 3\sqrt{2} \rfloor, \ldots\}$$

$$Spec(2 + \sqrt{2}) == \ \{3, 6, 10, 13, 17, 20, \cdots\}$$

Spectrum Observations

Observations

1. $Spec(\sqrt{2})$ and $Spec(2+\sqrt{2})$ are non-empty **sets**, not multisets

2. $Spec(\sqrt{2})$ and $Spec(2+\sqrt{2})$ don't seem to share any elements with each other

3. The set union of $Spec(\sqrt{2})$ and $Spec(2+\sqrt{2})$ seem to contain all of the natural numbers $n \geq 1$

This is interesting: if these properties are **proved** to be true then we can say that

$Spec(\sqrt{2})$ and $Spec(2+\sqrt{2})$ **form a partition** of the natural numbers $n \geq 1$

Spectrum Partition Theorem

More formally, for $Spec(\sqrt{2})$ and $Spec(2+\sqrt{2})$ to be a **partition** of the natural numbers greater equal 1, i.e. to be a **partition** of the set $N - \{0\}$ the following conditions must hold

Spectrum Partition Theorem

1. $Spec(\sqrt{2}) \neq \emptyset$ and $Spec(2+\sqrt{2}) \neq \emptyset$

2. $Spec(\sqrt{2}) \cap Spec(2+\sqrt{2}) = \emptyset$

3. $Spec(\sqrt{2}) \cup Spec(2+\sqrt{2}) = N - \{0\}$

The **proof** is not straight forward.

We first discuss a proof included in the **Book** and discuss its relationship to the Infinite Spectra

Finally we provide a correct proof

Finite Partition Theorem

First, we define certain **finite subsets** A_n, B_n of $Spec(\sqrt{2})$ and $Spec(2+\sqrt{2})$, respectively

Definition

$$A_n = \{m \in Spec(\sqrt{2}) : \quad m \leq n\}$$

$$B_n = \{m \in Spec(2+\sqrt{2}) \quad m \leq n\}$$

Remember

A_n and B_n are subsets of $\{1, 2, \ldots n\}$ for $n \in N - \{0\}$

Finite Partition Theorem

Given sets

$A_n = \{m \in Spec(\sqrt{2}) : \quad m \leq n\}$

$B_n = \{m \in Spec(2+\sqrt{2}) : \quad m \leq n\}$

Finite Spectrum Partition Theorem

1. $A_n \neq \emptyset$ and $B_n \neq \emptyset$
2. $A_n \cap B_n = \emptyset$
3. $A_n \cup B_n = \{1, 2, \ldots n\}$

Examples

We defined

$A_n = \{m \in Spec(\sqrt{2}) : \quad m \leq n\}$

$B_n = \{m \in Spec(2+\sqrt{2}) : \quad m \leq n\}$

Example $\quad n = 8$

We evaluate $\quad A_8 = \{1, 2, 4, 5, 7, 8\}, \quad B_8 = \{3, 6\}$

Observe that properties of the partition of the set $\{m \in Z^+ - \{0\} : \quad m \leq 8\}$ hold

1. $\quad A_8 \neq \emptyset$ and $B_8 \neq \emptyset$

2. $\quad A_8 \cap B_8 = \emptyset$

3. $\quad A_8 \cup B_8 = \{1, \cdots, 8\} = \{m \in N - \{0\} : \quad m \leq 8\}$

Observe that $\quad |A_8| + |B_8| = 8$

This property is an example of the general **property proved in the book**

Examples

We defined

$A_n = \{m \in Spec(\sqrt{2}) : \quad m \leq n\}$

$B_n = \{m \in Spec(2+\sqrt{2}) : \quad m \leq n\}$

Example $n = 15$

We evaluate

$A_{15} = \{1, 2, 4, 5, 7, 8, 9, 11, 12, 14, 15\}, \quad B_{15} = \{3, 6, 10, 13\}$

Again, that properties of the partition of the set $\{m \in N - \{0\} : \quad m \leq 15\}$ hold

1. $A_{15} \neq \emptyset$ and $B_{15} \neq \emptyset$

2. $A_{15} \cap B_{15} = \emptyset$

3. $A_{15} \cup B_{15} = \{1, \cdots, 15\} = \{m \in N - \{0\} : \quad m \leq 15\}$

Observe that $|A_{15}| + |B_{15}| = 15$

This property is again an example of the general **property proved in the book**

Finite Fact

Given sets

$A_n = \{m \in Spec(\sqrt{2}) : \quad m \leq n\}$

$B_n = \{m \in Spec(2 + \sqrt{2}) \quad m \leq n\}$

Finite Fact

For all $n \in N - \{0\}$

$$|A_n| + |B_n| = n$$

The book proves only this, and says that this is the **Spectrum Partition Theorem** for infinite Spectrum sets $Spec(\sqrt{2}),\ Spec(2 + \sqrt{2})$

Not so obvious!

Counting Elements

Before trying to prove the **Finite Fact** we first look for a closed formula to count the number of elements in subsets of a finite size of any spectrum

Given a spectrum $Spec(\alpha)$

Denote by $N(\alpha, n)$ the number of elements in the $Spec(\alpha)$ that are $\leq n$, i.e.

$$N(\alpha, n) = |\{m \in Spec(\alpha) : \quad m \leq n\}|$$

Counting Elements

We recall definition

$$Spec(\alpha) = \{\lfloor \alpha \rfloor, \lfloor 2\alpha \rfloor, \lfloor 3\alpha \rfloor, \cdots\}$$

We get immediately

$$m \in Spec(\alpha) \quad \text{iff} \quad m = \lfloor k\alpha \rfloor \quad \text{for} \quad \alpha \in R, \;\; k \in N - \{0\}$$

We re-write definition

$N(\alpha, n) = |\,\{m \in Spec(\alpha) : \quad m \leq n\}\,|$ as

$$N(\alpha, n) = |\,\{m : m = \lfloor k\alpha \rfloor \cap m \leq n \cap k > 0\}\,|$$

Hence

$$N(\alpha, n) = |\,\{\lfloor k\alpha \rfloor : \;\; \lfloor k\alpha \rfloor \leq n \;\cap\; k > 0\}\,| \quad n, k \in N - \{0\}$$

Counting Elements

We have

$N(\alpha, n) = \mid \{\lfloor k\alpha \rfloor : \; \lfloor k\alpha \rfloor \leq n \cap k > 0\} \mid$ for $n, k \in N - \{0\}$

Denote $P(k) \equiv \lfloor k\alpha \rfloor \leq n$ and $Q(k) \equiv k > 0$

We have that

$$N(\alpha, n) = \mid P(k) \cap Q(k) \mid$$

Recall re-write **Property 1** as two properties in a way we are going to use them

P1 $$\mid R(k) \mid = \sum_k [R(k)]$$

P2 $$\sum_k [R(k)] = \sum_{R(k)} 1 = \mid R(k) \mid$$

Counting Elements

We use property **P1** to $N(\alpha,n) = \mid P(k) \cap Q(k) \mid$ for $R(k) \equiv P(k) \cap Q(k)$ and we get

$$N(\alpha,n) = \mid P(k) \cap Q(k) \mid = \sum_k [P(k) \cap Q(k)]$$

Now we evaluate $N(\alpha,n)$ as follows

$$N(\alpha,n) = \sum_k [P(k)][Q(k)] = \sum_{Q(k)} [P(k)] = \sum_{k>0} [\lfloor k\alpha \rfloor \leq n]$$

We use now two known properties

$$m \leq n \text{ iff } m < n+1 \quad \text{and} \quad \lfloor x \rfloor < n \text{ iff } x < n$$

to transform $\lfloor k\alpha \rfloor \leq n$

Counting Elements

We have by the listed above properties

$$\lfloor k\alpha \rfloor \leq n \quad \text{iff} \quad \lfloor k\alpha \rfloor < n+1 \quad \text{iff} \quad k\alpha < n+1 \quad \text{iff} \quad k < \frac{n+1}{\alpha}$$

This justifies the following steps of computation

$$N(\alpha, n) = \sum_{k>0} [\lfloor k\alpha \rfloor \leq n] = \sum_{k>0} [\lfloor k\alpha < n+1] = \sum_{k>0} [k\alpha < n+1]$$

and we get

$$N(\alpha, n) = \sum_{k>0} \left[k < \frac{n+1}{\alpha} \right]$$

Counting Elements

We re-write the last sum using definition and property **P2**

$$N(\alpha,n) = \sum_{k>0}\left[k < \frac{n+1}{\alpha}\right] = \sum_{k}\left[k < \frac{n+1}{\alpha}\right][k>0]$$

$$= \sum_{k}\left[0 < k < \frac{n+1}{\alpha}\right] = \sum_{0<k<\frac{n+1}{\alpha}} 1$$

Using property **P2** again we get

$$N(\alpha,n) = \left| 0 < k < \frac{n+1}{\alpha} \right|$$

General Formula

Reminder $|\, 0 < k < \frac{n+1}{\alpha} \,|$ = number of integers in the interval $(0 \ldots \frac{n+1}{\alpha})$ and so by the the fact that interval $(\alpha \ldots \beta)$ has $\lceil \beta \rceil - \lceil \alpha \rceil - 1$ elements we evaluate

$$N(\alpha, n) = |\, 0 < k < \frac{n+1}{\alpha} \,| = \left\lceil \frac{n+1}{\alpha} \right\rceil - 0 - 1 = \left\lceil \frac{n+1}{\alpha} \right\rceil - 1$$

We have proved the following

General Formula

For any $\alpha \in R$ and a spectrum $Spec(\alpha)$ the number $N(\alpha, n)$ of elements in the $Spec(\alpha)$ that are $\leq n$ is given by the formula

$$N(\alpha, n) = \left\lceil \frac{n+1}{\alpha} \right\rceil - 1$$

Finite Fact

$$|A_n| + |B_n| = n \qquad \text{for any} \quad n \in N - \{0\}$$

where

$A_n = \{m \in Spec(\sqrt{2}) : \quad m \leq n\}$

$B_n = \{m \in Spec(2 + \sqrt{2}) : \quad m \leq n\}$

Proof

Observe that we defined $N(\alpha, n)$ as

$N(\alpha, n) = |\{m \in Spec(\alpha) : \quad m \leq n\}|$

and so we have that

$$|A_n| = N(\sqrt{2}, n) \quad \text{and} \quad |B_n| = N(2 + \sqrt{2},\ n)$$

We hence have to prove that

$$N(\sqrt{2},\ n) + N(2 + \sqrt{2},\ n) = n$$

Finite Fact Proof

We use the **General Formula** $N(\alpha, n) = \lceil \frac{n+1}{\alpha} \rceil - 1$ for $\alpha_1 = \sqrt{2}$ and $\alpha_2 = 2 + \sqrt{2}$ and evaluate by using property $\lceil x \rceil - 1 = \lfloor x \rfloor$ for $x \notin Z$

$$N(\alpha_1, n) + N(\alpha_2, n)) = \left\lceil \frac{n+1}{\sqrt{2}} \right\rceil - 1 + \left\lceil \frac{n+1}{2+\sqrt{2}} \right\rceil - 1$$

$$= \left\lfloor \frac{n+1}{\sqrt{2}} \right\rfloor + \left\lfloor \frac{n+1}{2+\sqrt{2}} \right\rfloor$$

Now we use property $\lfloor x \rfloor = x - \{x\}$, where $\{x\}$ is a **fractional** part of x and get

$$N(\alpha_1, n) + N(\alpha_2, n)) = \frac{n+1}{\sqrt{2}} - \left\{ \frac{n+1}{\sqrt{2}} \right\} + \frac{n+1}{2+\sqrt{2}} - \left\{ \frac{n+1}{2+\sqrt{2}} \right\}$$

Finite Fact Proof

We continue evaluation using identity $\frac{1}{\sqrt{2}} + \frac{1}{2+\sqrt{2}} = 1$

$$N(\alpha_1, n) + N(\alpha_2, n)) = \frac{n+1}{\sqrt{2}} + \frac{n+1}{2+\sqrt{2}} - \left\{\frac{n+1}{\sqrt{2}}\right\} - \left\{\frac{n+1}{2+\sqrt{2}}\right\}$$

$$= (n+1)\left(\frac{1}{\sqrt{2}} + \frac{1}{2+\sqrt{2}}\right) - \left(\left\{\frac{n+1}{\sqrt{2}}\right\} + \left\{\frac{n+1}{2+\sqrt{2}}\right\}\right)$$

$$= (n+1) - \left(\left\{\frac{n+1}{\sqrt{2}}\right\} + \left\{\frac{n+1}{2+\sqrt{2}}\right\}\right)$$

Observe that if we show that $\left\{\frac{n+1}{\sqrt{2}}\right\} + \left\{\frac{n+1}{2+\sqrt{2}}\right\} = 1$

then we have succeeded to prove the **Finite Fact**

Finite Fact Proof

We have proved as a part of our computations that

$$\frac{n+1}{\sqrt{2}} + \frac{n+1}{2+\sqrt{2}} = n+1$$

and now we can use it to prove

$$\left\{\frac{n+1}{\sqrt{2}}\right\} + \left\{\frac{n+1}{2+\sqrt{2}}\right\} = 1$$

We prove more general **Special Property** and get our property as a particular case

Special Property Proof

Special Property

For any $x_1, x_2 \notin Z$

$$\text{If} \quad x_1 + x_2 = n + 1 \quad \text{then} \quad \{x_1\} + \{x_2\} = 1$$

Proof

Let $x_1 = \lfloor x_1 \rfloor + \{x_1\}$ and $x_2 = \lfloor x_2 \rfloor + \{x_2\}$

Assume that

$$x_1 + x_2 = \lfloor x_1 \rfloor + \{x_1\} + \lfloor x_2 \rfloor + \{x_2\} = n + 1$$

Since $x_1, x_2 \notin Z$ we get that $\{x_1\} \neq 0$, $\{x_2\} \neq 0$ and so

$$0 < \{x_1\} < 1 \quad \text{and} \quad 0 < \{x_2\} < 1$$

Adding the above inequalities we get

$$0 < \{x_1\} + \{x_2\} < 2$$

Special Property Proof

Observe that $\lfloor x_1 \rfloor + \lfloor x_2 \rfloor = m \in Z$

Denote $\{x_1\} + \{x_2\} = \theta$

We assumed

$$n+1 = \lfloor x_1 \rfloor + \{x_1\} + \lfloor x_2 \rfloor + \{x_2\}$$

so we have

$$n+1 = m+\theta \quad \text{for} \quad 0 < \theta < 2 \text{ and } m \in Z$$

Hence it must be that $\theta \in Z$

But $0 < \theta < 2$ and it is possible only when $\theta = 1$, i.e. $\{x_1\} + \{x_2\} = 1$

This ends the proof

Finite Fact

Put $x_1 = \frac{n+1}{\sqrt{2}}, \quad x_2 = \frac{n+1}{2+\sqrt{2}}$

By Special Property we have that

$$\left\{\frac{n+1}{\sqrt{2}}\right\} + \left\{\frac{n+1}{2+\sqrt{2}}\right\} = 1$$

It ends the proof of our

Finite Fact

$$|A_n| + |B_n| = n \qquad \text{for any} \quad n \in N - \{0\}$$

where

$A_n = \{m \in Spec(\sqrt{2}) : \quad m \leq n\}$

$B_n = \{m \in Spec(2+\sqrt{2}) : \quad m \leq n\}$

Book Statement

The Book proves the **Finite Fact** and states on page 78
" A PARTITION IT IS"

The meaning of this is that the **Finite Fact** implies obviously without any additional proof the following

Spectrum Partition Theorem

1. $Spec(\sqrt{2}) \neq \emptyset$ and $Spec(2+\sqrt{2}) \neq \emptyset$

2. $Spec(\sqrt{2}) \cap Spec(2+\sqrt{2}) = \emptyset$

3. $Spec(\sqrt{2}) \cup Spec(2+\sqrt{2}) = N - \{0\}$

We are going to show now that it is not so obvious even in the case of **Finite** Spectrum Partition

The **infinite** case will be discussed after

Let's analyze what we have!

Finite Spectrum Partition

Given sets

$A_n = \{m \in Spec(\sqrt{2}) : \quad m \leq n\}$

$B_n = \{m \in Spec(2 + \sqrt{2}) : \quad m \leq n\}$

Finite Spectrum Partition Theorem - to be proved

1. $A_n \neq \emptyset$ and $B_n \neq \emptyset$

2. $A_n \cap B_n = \emptyset$

3. $A_n \cup B_n = \{1, 2, \ldots n\}$

Finite Fact - just proved

$$|A_n| + |B_n| = n \qquad \text{for any} \quad n \in N - \{0\}$$

Question Is it possible to **prove** Finite Spectrum Partition Theorem from the Finite Fact?

Finite Partition

Definition Finite Partition

Let X be a **non-empty, finite** set; i.e $X \neq \emptyset$ and $|X| = n$ for some $n \in N - \{0\}$

We say that sets $A, B \subseteq X$ such that $A \neq B$ form a **finite partition** of the set X when the following conditions are satisfied

1. $A \neq \emptyset$ and $B \neq \emptyset$

2. $A \cap B = \emptyset$

3. $A \cup B = X$

Sets Finite Fact $|A| + |B| = |X|$

When $|X| = n$ we write it as $|A| + |B| = n$

Let's now examine the relationship between the Finite Partition and Sets Finite Fact

Finite Partition and Sets Finite Fact

We show now that the Finite Partition **implies** the **Sets Finite Fact**, i.e. we prove the following

Fact P1

If sets A, B form a finite partition of the finite set X, **then** $|A| + |B| = |X|$

Proof

Assume that A, B form a finite partition then by condition **1.** and **3.** $A \cup B = X$, $A \neq \emptyset$ and $B \neq \emptyset$

So $|A \cup B| = |X|$ and $|X| \geq 1$

The sets A, B are finite, hence

$$|A \cup B| = |A| + |B| - |A \cap B|$$

but by **2.** $A \cap B = \emptyset$ and so $|A \cap B| = 0$ and $|A \cup B| = |A| + |B|$ and as $|A \cup B| = |X|$ we have that

$$|A| + |B| = |X|$$

Counter-Examples

We show now that the Sets Finite Fact **does not always imply** the Finite Partition, i.e. we give the following following counter-examples covering all cases

Counter-Example 1

Take the sets $X = \{1,2,3,4\}$, $A = \{2\}$, $B = \{1,\ 2,\ 3\}$

We have that

$$|A| + |B| = 1 + 3 = 4 = |X| \quad \text{and} \quad A \cap B = \{2\} \neq \emptyset$$

and condition **2.** of Finite Partition **does not hold**

Counter-Examples

Counter-Example 2

We also have for the same sets
$X = \{1, 2, 3, 4\}, \ \ A = \{2\}, \ \ B = \{1,\ 2,\ 3\}$ that the condition **3.** of Finite Partition **does not hold** as

$$|A| + |B| = 4 = |X| \quad \text{and} \quad A \cup B = \{1, 2, 3\} \neq X$$

Counter-Example 3 Take the sets
$X = \{1\}, \ \ A = \{1\}, \ \ B = \emptyset$, or $\ B = \{1\}, \ \ A = \emptyset$

We have that

$$|A| + |B| = 1 = |X| \quad \text{and} \quad A = \emptyset \ \text{ or } \ B = \emptyset$$

and condition **1.** of Finite Partition **does not hold**

Useful Facts

We are going to prove two useful facts that relate to our

Question Is it possible to **prove** Finite Spectrum Partition Theorem from the Sets Finite Fact?

Fact P2

If $|A| + |B| = |X|$ and $A \neq \emptyset$, $B \neq \emptyset$ and $A \cap B = \emptyset$

then the sets A, B form a **finite partition** of X

Proof

We prove the condition **3.** by contradiction

Let $|A| + |B| = |X|$ and $A \cup B \neq X$, i.e. $|A \cup B| \neq |X|$

We evaluate

$|A \cup B| = |X| = |A| + |B| - |A \cap B| = |A| + |B|$ and get a **contradiction**

$$|A \cup B| = |X| \quad \text{and} \quad |A \cup B| \neq |X|$$

Useful Facts

Fact P3

If $|A| + |B| = |X|$ and $A \neq \emptyset$, $B \neq \emptyset$ and $A \cup B = X$
then the sets A, B form a **finite partition** of the set X

Proof

We prove the condition **2.**

Let $|A| + |B| = |X|$ and $A \cup B = X$, i.e. $|A \cup B| = |X|$

We evaluate

$$|A \cup B| = |X| = |A| + |B| - |A \cap B| = |A| + |B|$$

and

$$|A| + |B| - |A \cap B| = |A| + |B| \quad \text{iff} \quad A \cap B = \emptyset$$

This proves that the condition **2.** **holds**

Back to Finite Spectrum Partition Theorem

Facts **P2**, and **P3** say:

if the sets A, B are non-empty, disjoint, or $A \cup B = X$ **then**
Finite Fact implies Finite Partition

Take now

$$X = \{1, 2 \ldots n\}, \quad A = A_n, \quad B = B_n$$

The Finite Partition becomes

Finite Spectrum Partition Theorem

1. $A_n \neq \emptyset$ and $B_n \neq \emptyset$

2. $A_n \cap B_n = \emptyset$

3. $A_n \cup B_n = \{1, 2, \ldots n\}$

Question and Answers

The Sets Finite Fact becomes

Finite Fact $|A_n| + |B_n| = n$, for $n \in N - \{0\}$

We are now ready to answer our

Question Does the Sets Finite Fact **implies** as the Book states, the Finite Spectrum Partition Theorem?

Answer YES, but only under conditions specified in the Facts **P2**, and **P3**

Question and Answers

Observe that $A_n \neq \emptyset$ and $B_n \neq \emptyset$

Hence, by the **Fact P2** we have to **prove** that

$$A_n \cap B_n = \emptyset$$

in order to have that the Finite Spectrum Partition Theorem holds

or by the **Fact P2** we have to prove that

$$A_n \cup B_n = \{1, 2, \ldots n\}$$

We now **choose** to to use **Fact P2** and to prove that $A_n \cap B_n = \emptyset$

Spectrum Fact

Reminder

$$A_n \subseteq Spec(\sqrt{2}) \quad \text{and} \quad B_n \subseteq Spec(2+\sqrt{2})$$

We hence prove now a more general fact (always do it when you can!)

Spectrum Fact

$$Spec(\sqrt{2}) \cap Spec(2+\sqrt{2}) = \emptyset$$

We recall definition

$$Spec(\alpha) = \{\lfloor \alpha \rfloor, \lfloor 2\alpha \rfloor, \lfloor 3\alpha \rfloor, \cdots\}$$

We get immediately

$$m \in Spec(\alpha) \quad \text{iff} \quad m = \lfloor k\alpha \rfloor$$

Spectrum Fact Proof

Proof

We prove this fact by contradiction

Assume that $Spec(\sqrt{2}) \cap Spec(2+\sqrt{2}) \neq \emptyset$

By definition it means that there is $n \in N - \{0\}$ such that

$$n \in Spec(\sqrt{2}) \quad \text{and} \quad n \in Spec(2+\sqrt{2})$$

i.e. there are $k_1, k_2 \in N - \{0\}$ such that

$$n = \lfloor k_1\sqrt{2} \rfloor \quad \text{and} \quad n = \lfloor k_2(2+\sqrt{2}) \rfloor$$

We use now property

8. $\lfloor x \rfloor = n$ if and only if $n \leq x < n+1$ for $x \in R$, $n \in Z$

Spectrum Fact Proof

By **8.** convert these two equalities to two inequalities

$$n \leq \quad k_1\sqrt{2} \quad < n+1 \tag{1}$$
$$n \leq \quad k_2(2+\sqrt{2}) \quad < n+1 \tag{2}$$

Now we can drop the equality condition in the inequalities (1) and (2) because $n \in N - \{0\}$, but $k_1\sqrt{2}$ and $k_2(2+\sqrt{2})$ are two irrational numbers

Thus we get

$$n < \quad k_1\sqrt{2} \quad < n+1 \tag{3}$$
$$n < \quad k_2(2+\sqrt{2}) \quad < n+1 \tag{4}$$

Spectrum Fact Proof

We divide (3) by $\sqrt{2}$ and (4) by $k_2(2+\sqrt{2})$

$$\frac{n}{\sqrt{2}} < k_1 < \frac{n+1}{\sqrt{2}} \tag{5}$$

$$\frac{n}{2+\sqrt{2}} < k_2 < \frac{n+1}{2+\sqrt{2}} \tag{6}$$

Now we add (5) and (6) together, to get:

$$\frac{n}{\sqrt{2}} + \frac{n}{2+\sqrt{2}} < k_1 + k_2 < \frac{n+1}{\sqrt{2}} + \frac{n+1}{2+\sqrt{2}}$$

Grouping for n and $n+1$

$$n\left(\frac{1}{\sqrt{2}} + \frac{1}{2+\sqrt{2}}\right) < k_1 + k_2 < (n+1)\left(\frac{1}{\sqrt{2}} + \frac{1}{2+\sqrt{2}}\right)$$

Spectrum Fact Proof

The two factors for n and $n+1$ are equal

Let's evaluate them

$$\frac{1}{\sqrt{2}} + \frac{1}{2+\sqrt{2}} = \frac{2+2\sqrt{2}}{\sqrt{2}(2+\sqrt{2})} = \frac{2+2\sqrt{2}}{2\sqrt{2}+\sqrt{2}\sqrt{2}} = \frac{2+2\sqrt{2}}{2\sqrt{2}+2} = 1$$

This simplifies our inequality to

$$n < k_1 + k_2 < n+1$$

But this is a **contradiction**:

n and $n+1$ are two **consecutive** integers, so no other integer $k_1 + k_2$ can belong to the interval

Finite Spectrum Partition Theorem

We get as a collolary that $A_n \cap B_n = \emptyset$

We have hence by **Fact P2** finally proved the

Finite Spectrum Partition Theorem

1. $A_n \neq \emptyset$ and $B_n \neq \emptyset$

2. $A_n \cap B_n = \emptyset$

3. $A_n \cup B_n = \{1, 2, \ldots n\}$

It was a LONG WAY! but we are **not finished** yet!

All we got is the Finite Spectrum Partition Theorem not the ”full” Spectrum Partition Theorem

Spectrum Partition Theorem Proof

Spectrum Partition Theorem

1. $Spec(\sqrt{2}) \neq \emptyset$ and $Spec(2+\sqrt{2}) \neq \emptyset$

2. $Spec(\sqrt{2}) \cap Spec(2+\sqrt{2}) = \emptyset$

3. $Spec(\sqrt{2}) \cup Spec(2+\sqrt{2}) = N - \{0\}$

Proof

1. holds by definition of the spectrum, as always
$\lfloor \alpha \rfloor \in Spec(\alpha) \lfloor \alpha \rfloor$

2. holds by just proved **Spectrum Fact**

3. - the proof follows

Observe that

$$\textbf{S} \quad Spec(\sqrt{2}) = \bigcup_{n \geq 1} A_n \quad \text{and} \quad Spec(2+\sqrt{2}) = \bigcup_{n \geq 1} B_n$$

Spectrum Partition Theorem Proof

From the Finite Spectrum Partition Theorem we have that for all $n \in N - \{0\}$

$$A_n \cup B_n = \{1, 2, \ldots n\}$$

Hence by

$$\bigcup_{n \geq 1} (A_n \cup B_n) = \bigcup_{n \geq 1} \{1, 2, \ldots n\} = N - \{0\}$$

But by above the general sums distributivity law we get the following

$$\bigcup_{n \geq 1} (A_n \cup B_n) = \bigcup_{n \geq 1} A_n \cup \bigcup_{n \geq 1} B_n = N - \{0\}$$

Spectrum Partition Theorem Proof

But by definition **S**

S $Spec(\sqrt{2}) = \bigcup_{n \geq 1} A_n$ and $Spec(2 + \sqrt{2}) = \bigcup_{n \geq 1} B_n$

we get

$$Spec(\sqrt{2}) \cup Spec(2 + \sqrt{2}) = N - \{0\}$$

THIS ENDS THE PFOOF!!

General Spectrum Partition Theorem

We are going now to give a proof of our Spectrum Partition Theorem that is independent of the BOOK

It is simple and elegant and . . . does not use the SUMS!

Do do so, we GENERALIZE the problem a bit, prove the generalization and get our Theorem as a particular case

Here it is!

Generalization

General Spectrum Partition Theorem

Let $\alpha > 0,\ \beta > 0,\quad \alpha,\ \beta \in R - Q$ be such that

$$\frac{1}{\alpha} + \frac{1}{\beta} = 1$$

Then the sets

$$A = \{\lfloor n\alpha \rfloor :\quad n \in N - \{0\}\} = Spec(\alpha)$$

$$B = \{\lfloor n\beta \rfloor :\quad n \in N - \{0\}\} = Spec(\beta)$$

form a **partition** of $Z^+ = N - \{0\}$, i.e.

1. $A \neq \emptyset$ and $B \neq \emptyset$
2. $A \cap B = \emptyset$
3. $A \cup B = Z^+$

Proof

1. $A \neq \emptyset$ and $B \neq \emptyset$ holds as $\lfloor \alpha \rfloor \in A$ and $\lfloor \beta \rfloor \in B$

We prove this fact by **contradiction**

Assume that $A \cap B \neq \emptyset$

By definition it means that there is $k \in Z^+$ such that

$$k \in A \quad \text{and} \quad k \in B$$

i.e. there are $i, j \in Z^+$ such that

$$k = \lfloor i\alpha \rfloor \quad \text{and} \quad k = \lfloor j\beta \rfloor$$

We use now property

8. $\lfloor x \rfloor = k$ if and only if $k \leq x < k+1$ for $x \in R$, $n \in Z^+$

Proof

By **8.** convert these two equalities to two inequalities

$$k \le \quad i\alpha \quad < k+1 \tag{7}$$
$$k \le \quad j\beta \quad < k+1 \tag{8}$$

Now we can drop the equality condition in the inequalities (7) and (8) because $k \in Z^+$, but $\alpha,\ \beta \in R - Q$, so $i\alpha,\ j\beta$ can't be integers

Thus we get

$$k < \quad i\alpha \quad < k+1 \tag{9}$$
$$k < \quad j\beta \quad < k+1 \tag{10}$$

Proof

We divide (9) by α and (10) by β- we can do it as $\alpha > 0,\ \beta > 0$ and we get

$$\frac{k}{\alpha} < \quad i \quad < \frac{k+1}{\alpha} \qquad (11)$$

$$\frac{k}{\beta} < \quad j \quad < \frac{k+1}{\beta} \qquad (12)$$

Now we add (11) and (12) together, to get:

$$\frac{k}{\alpha} + \frac{k}{\beta} < i + j < \frac{k+1}{\alpha} + \frac{k+1}{\beta}$$

Grouping for k and $k+1$

$$k\left(\frac{1}{\alpha} + \frac{1}{\beta}\right) < i + j < (k+1)\left(\frac{1}{\alpha} + \frac{1}{\beta}\right)$$

Proof

The two factors for k and $k+1$ are equal by the Theorem **assumption**

$$\frac{1}{\alpha}+\frac{1}{\beta}=1$$

This simplifies our inequality to

$$k < i+j < k+1$$

But this is a **contradiction**:

k and $k+1$ are two **consecutive** positive integers, so no other positive integer $i+j$ can belong to the interval

Haven't you seen a similar proof before???

Proof

Now as the last step we prove

3. $A \cup B = Z^+$

We carry proof by **contradiction**

Assume that $A \cup B \neq Z^+$

It means that there is $k \in Z^+$ such that

$$k \notin A \quad \text{and} \quad k \notin B$$

By definition of sets A, B we have

$$k \notin A \quad \text{iff} \quad k \neq \lfloor n\alpha \rfloor \quad \text{for all} \quad n \in Z^+$$

$$k \notin B \quad \text{iff} \quad k \neq \lfloor n\beta \rfloor \quad \text{for all} \quad n \in Z^+$$

Proof

Observe that if $k \neq \lfloor n\alpha \rfloor$ **for all** $n \in Z^+$, then as $\lfloor n\alpha \rfloor \neq k$, $\lfloor (n+1)\alpha \rfloor \neq k$, and $\lfloor n\alpha \rfloor < \lfloor (n+1)\alpha \rfloor$ **there exist** $i_0, j_0 \in Z^+$ such that

$$(\star) \quad \lfloor i_0\alpha \rfloor < k \quad \text{and} \quad \lfloor (i_0+1)\alpha \rfloor \geq k+1$$

and similarly

$$(\star\star) \quad \lfloor j_0\beta \rfloor < k \quad \text{and} \quad \lfloor (j_0+1)\beta \rfloor \geq k+1$$

We now transform $(\star)$ and $(\star\star)$ by using he properties

13. $\lfloor x \rfloor < n$ if and only if $x < n$

16. $x \geq \lfloor n \rfloor$ if and only if $x \geq n$

Proof

Now we can drop the equality condition applying the inequality **16.** because with $k \in Z^+$ and $\alpha,\ \beta \in R - Q$, we have that $(i_0+1)\alpha,\ (j_0+1)\beta$ can't be integers

We get hence that

$$(1) \quad i_0\alpha < k \quad \text{and} \quad (i_0+1)\alpha > k+1$$

$$(2) \quad j_0\beta < k \quad \text{and} \quad (j_0+1)\beta > k+1$$

We re-write (1), (2) respectively as follows

$$\alpha < \frac{k}{i_0} \quad \text{and} \quad \alpha > \frac{k+1}{(i_0+1)}$$

$$\beta < \frac{k}{j_0} \quad \text{and} \quad \beta > \frac{k+1}{(j_0+1)}$$

Proof

We know that for any $a, b \in Z^+$,

$$a < b \quad \text{iff} \quad \frac{1}{a} > \frac{1}{b}$$

We hence re-write (1), (2) further as

$$\frac{1}{\alpha} > \frac{i_0}{k} \quad \text{and} \quad \frac{1}{\alpha} < \frac{i_0+1}{k+1}$$

i.e

$$(3) \quad \frac{i_0}{k} < \frac{1}{\alpha} < \frac{i_0+1}{k+1}$$

and similarly we get

$$(4) \quad \frac{j_0}{k} < \frac{1}{\beta} < \frac{j_0+1}{k+1}$$

Proof

Adding (3) and (4) and using the **assumption**

$$\frac{1}{\alpha} + \frac{1}{\beta} = 1$$

we get that

$$\frac{i_0 + j_0}{k} < 1 < \frac{i_0 + j_0 + 2}{k + 1}$$

This is equivalent to

$$\frac{i_0 + j_0}{k} < 1 \quad \text{and} \quad 1 < \frac{i_0 + j_0 + 2}{k + 1}$$

$$i_0 + j_0 < k \quad \text{and} \quad k + 1 < i_0 + j_0 + 2$$

Hence

$$i_0 + j_0 < k < i_0 + j_0 + 1$$

Contradiction! as $i_0, j_0, k \in Z^+$

This ends the proof

Floor and Ceilings Sums

Example Evaluate

$$\sum_{0 \le k < n} \lfloor \sqrt{k} \rfloor$$

Hint: use

$$\sum_{0 \le k < n} \lfloor \sqrt{k} \rfloor = \sum_{0 \le k < n} \sum_{m \ge 0,\ m = \lfloor \sqrt{k} \rfloor} m$$

We evaluate

$$\sum_{0 \le k < n} \lfloor \sqrt{k} \rfloor = \sum_{0 \le k < n} \sum_{m \ge 0} m\,[m = \lfloor \sqrt{k} \rfloor]$$

$$= \sum_{m \ge 0} \sum_{k \ge 0} m\,[k < n][m = \lfloor \sqrt{k} \rfloor]$$

Floor and Ceilings Sums

We use now property and get

8. $\lfloor x \rfloor = n$ if and only if $n \leq x < n+1$

and we get

$$\sum_{0 \leq k < n} \lfloor \sqrt{k} \rfloor = \sum_{m \geq 0,\ k \geq 0} m[k < n][m \leq \sqrt{k} < m+1]$$

$$= \sum_{m \geq 0,\ k \geq 0} m[k < n \cap m^2 \leq k < (m+1)^2]$$

Let's look now at

$$P(k,m,n) \equiv k < n \cap m^2 \leq k < (m+1)^2$$

Floor and Ceilings Sums

We evaluate $P(k,m,n) \equiv k < n \cap m^2 \leq k < (m+1)^2$

$$\equiv m^2 \leq k < n < (m+1)^2 \cup m^2 \leq k < (m+1)^2 \leq n$$

i.e. $P(k,m,n) \equiv Q \cup R$ and we know that

$$\sum_{m,k} [Q \cup R] = \sum_{m,k} [Q] + \sum_{m,k} [R] - \sum_{m,k} [Q \cap R]$$

and here $Q \cap R$ is false, i.e. $\sum_{m,k} [Q \cap R] = 0$ and we get

$$\sum_{0 \leq k < n} \lfloor \sqrt{k} \rfloor = \sum_{m,k \geq 0} m \, [m^2 \leq k < n < (m+1)^2]$$

$$+ \sum_{m,k \geq 0} m \, [m^2 \leq k < (m+1)^2 \leq n]$$

Floor and Ceilings Sums

Assume now $n = a^2$ for certain $a \in N$, i.e. n is a perfect square

The first sum becomes

$$\sum_{m,k \geq 0} m\,[m^2 \leq k < a^2 < (m+1)^2] = 0$$

because the statement

$$m^2 \leq k < a^2 < (m+1)^2$$

is FALSE as there is no $a \in N$ such that $m < a < m+1$

Floor and Ceilings Sums

We proved that

$$\sum_{0\le k<n} \lfloor \sqrt{k} \rfloor = \sum_{m,k\ge 0} m\,[m^2 \le k < (m+1)^2 \le a^2]$$

Evaluate now

$$m^2 \le k < (m+1)^2 \le a \;\equiv\; m^2 \le k < (m+1)^2 \cap (m+1)^2 \le a^2$$

$$\equiv\; m^2 \le k < (m+1)^2 \cap (m+1) \le a$$

$$\sum_{0\le k<n} \lfloor \sqrt{k} \rfloor = \sum_{m,k\ge 0} m\,[m^2 \le k < (m+1)^2]\,[(m+1) \le a]$$

Floor and Ceilings Sums

We evaluate

$$\sum_{m,k\geq 0} m\,[m^2 \leq k < (m+1)^2]\,[(m+1) \leq a]$$

$$= \sum_{m\geq 0} \sum_{k\geq 0} m\,[(m+1) \leq a]\,[m^2 \leq k < (m+1)^2]$$

$$= \sum_{m\geq 0} m\,[(m+1) \leq a] \sum_{k\geq 0} [m^2 \leq k < (m+1)^2]$$

$$= \sum_{m\geq 0} m\,[(m+1) \leq a] \sum_{k\geq 0} [k \in [m^2 \ldots (m+1)^2]$$

Floor and Ceilings Sums

We recall the properties

$$\sum_k [R(k)] = \sum_{R(k)} 1 = | R(k) |$$

$$[\alpha...\beta) \quad \text{contains exactly} \quad \lceil\beta\rceil - \lceil\alpha\rceil \quad \text{integers}$$

and get

$$\sum_{k\geq 0} [k \in [m^2 \dots (m+1)^2] = 2m+1$$

Hence

$$\sum_{m\geq 0} m\, [(m+1) \leq a] \sum_{k\geq 0} [k \in [m^2 \dots (m+1)^2]$$

$$= \sum_{m\geq 0} m(2m+1)\, [(m+1) \leq a] = \sum_{m\geq 0} (2m^2+m)\, [(m+1) \leq a]$$

Floor and Ceilings Sums

We have hence proved that

$$\sum_{0\le k<n} \lfloor \sqrt{k} \rfloor = \sum_{m\ge 0} (2m^2+m)\ [(m+1)\le a]$$

Recall that $x^{\underline{2}} = x(x-1) = x^2 - x$ and $x^{\underline{1}} = x$

Evaluate

$$2m^2+m = 2m^2-2m+2m+m = 2m(m-1)+3m = 2m^{\underline{2}}+3m^{\underline{1}}$$

Also we have that $m+1 \le a$ iff $m < a$, so now

$$\sum_{0\le k<n} \lfloor \sqrt{k} \rfloor = \sum_{m\ge 0} (2m^{\underline{2}}+3m^{\underline{1}})[m<a]$$

Floor and Ceilings Sums

Last steps

$$\sum_{m\geq 0}(2m^{\underline{2}}+3m^{\underline{1}})[m<a]=\sum_{0\leq m<a}(2m^{\underline{2}}+3m^{\underline{1}})$$

$$=\sum\nolimits_0^a(2m^{\underline{2}}+3m^{\underline{1}})\delta m=(2\frac{m^{\underline{3}}}{3}+3\frac{m^{\underline{2}}}{2})\Big|_0^a$$

$$=\frac{2}{3}m(m-1)(m-2)+\frac{3}{2}m(m-1)\Big|_0^a=\frac{1}{6}(a-1)a(a+1)$$

and

$$\sum_{0\leq k<n}\lfloor\sqrt{k}\rfloor=\frac{1}{6}(a-1)a(a+1)$$

Homework: do the case (page 87) $a=\lfloor\sqrt{k}\rfloor$

END of CHAPTER 3