cse547, math547
DISCRETE MATHEMATICS

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LECTURE 11

CHAPTER 3
INTEGER FUNCTIONS

PART1: Floors and Ceilings

PART 2: Floors and Ceilings Applications

PART 1
Floors and Ceilings

Floor and Ceiling Definitions

Floor Definition

For any $x \in R$ we define

$\lfloor x \rfloor$ = the greatest integer less than or equal to x

Ceiling Definition

For any $x \in R$ we define

$\lceil x \rceil$ = the least (smallest) integer greater than or equal to x

Floor and Ceiling Definitions

Definitions written Symbolicaly

Floor

$$\lfloor x \rfloor = max\{a \in Z : \quad a \leq x\}$$

Ceiling

$$\lceil x \rceil = min\{a \in Z : \quad a \geq x\}$$

Floor and Ceiling Basics

Remark: we use, after the book the notion of max, min elements instead of the least(smallest) and greatest elements because for the Posets P_1, P_2 we have that

$P_1=(\{ a\in Z : a \leq x \},\leq)$ has unique max element that is the greatest and

$P_2=(\{ a\in Z : a \geq x \},\geq)$ has unique min element that is the least (smallest)

Floor and Ceiling Basics

Fact 1

For any $x \in R$

$\lfloor x \rfloor$ and $\lceil x \rceil$ **exist** and are **unique**

We **define** functions

Floor

$$f_1 : \quad R \longrightarrow Z$$

$$f_1(x) = \lfloor x \rfloor = max\{a \in Z : \quad a \leq x\}$$

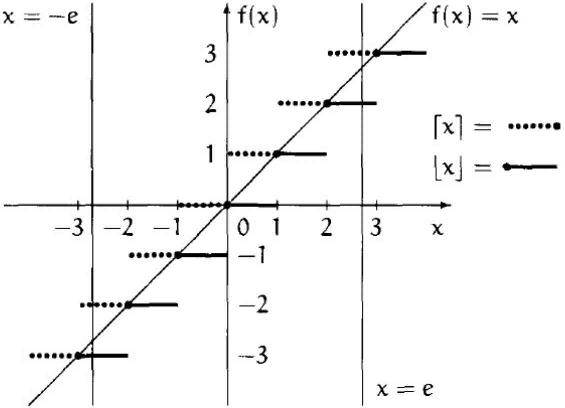
Ceiling

$$f_2 : \quad R \longrightarrow Z$$

$$f_2(x) = \lceil x \rceil = min\{a \in Z : \quad a \geq x\}$$

Floor and Ceiling Basics

Graphs of f_1, f_2



Properties of $\lfloor x \rfloor$ and $\lceil x \rceil$

1. $\lfloor x \rfloor = x$ if and only if $x \in Z$

2. $\lceil x \rceil = x$ if and only if $x \in Z$

3. $x - 1 < \lfloor x \rfloor \leq \lceil x \rceil < x + 1 \quad x \in R$

4. $\lfloor -x \rfloor = -\lceil x \rceil \quad x \in R$

Properties of $\lfloor x \rfloor$ and $\lceil x \rceil$

5. $\lceil -x \rceil = -\lfloor x \rfloor \quad x \in R$

6. $\lceil x \rceil - \lfloor x \rfloor = [x \notin Z]$ characteristic function of $x \notin Z$

we re- write **6.** as follows

7. $\lceil x \rceil - \lfloor x \rfloor = 0$ for $x \in Z$

$\lceil x \rceil - \lfloor x \rfloor = 1$ for $x \notin Z$

Properties of $\lfloor x \rfloor$ and $\lceil x \rceil$

8. $\lfloor x \rfloor = n$ if and only if $n \le x < n+1$ for $x \in R$, $n \in Z$

9. $\lfloor x \rfloor = n$ if and only if $x - 1 < n \le x$ for $x \in R$, $n \in Z$

Properties of $\lfloor x \rfloor$ and $\lceil x \rceil$

10. $\lceil x \rceil = n$ if and only if $n-1 < x \leq n$ for $x \in R$, $n \in Z$

11. $\lceil x \rceil = n$ if and only if $x \leq n < x+1$ for $x \in R$, $n \in Z$

12. $\lfloor x+n \rfloor = \lfloor x \rfloor + n$ and $\lceil x+n \rceil = \lceil x \rceil + n$ for $x \in R$, $n \in Z$

Some Proofs

Proof of

12. $\lfloor x+n \rfloor = \lfloor x \rfloor + n$ for $x \in R$, $n \in Z$

Directly from definition we have that

$$\lfloor x \rfloor \leq x < \lfloor x \rfloor + 1$$

Adding n to all sides we get

$$\lfloor x \rfloor + n \leq x + n < \lfloor x \rfloor + n + 1$$

Applying

8. $\lfloor x \rfloor = m$ if and only if $m \leq x < m+1$ for $x \in R$, $m \in Z$

for $m = \lfloor x \rfloor + n$ we get $\lfloor x+n \rfloor = m$, i.e.

$$\lfloor x+n \rfloor = \lfloor x \rfloor + n$$

Some Proofs

Observe that it is **not true** that for all $x \in R,\ n \in Z$

$$\lfloor nx \rfloor = n\lfloor x \rfloor$$

Take $n = 2,\ x = \frac{1}{2}$ and we get that

$$\left\lfloor 2 \cdot \frac{1}{2} \right\rfloor = 1 \neq 2\left\lfloor \frac{1}{2} \right\rfloor = 0$$

More Properties of $\lfloor x \rfloor$ and $\lceil x \rceil$

In all properties $x \in R$, $n \in Z$

13. $x < n$ if and only if $\lfloor x \rfloor < n$

14. $n < x$ if and only if $n < \lceil x \rceil$

15. $x \leq n$ if and only if $\lceil x \rceil \leq n$

16. $n \leq x$ if and only if $n \leq \lfloor x \rfloor$

Some Proofs

Proof of **13.** $x < n$ if and only if $\lfloor x \rfloor < n$

Let $x < n$

We know that $\lfloor x \rfloor \leq$ x so $\lfloor x \rfloor \leq x < n$

and hence $\lfloor x \rfloor <$ n

Let $\lfloor x \rfloor <$ n

By property **3.** $x - 1 < \lfloor x \rfloor \leq \lceil x \rceil < x + 1, \quad x \in R$

$x - 1 < \lfloor x \rfloor$, i.e $x < \lfloor x \rfloor + 1$

But $\lfloor x \rfloor < n$, so $\lfloor x \rfloor + 1 \leq n$ and

$$x < \lfloor x \rfloor + 1 \leq n$$

Hence $x < n$ what ends the proof

Fractional Part of x

Definition

We define: $\{x\} = x - \lfloor x \rfloor$

$\{x\}$ is called a **fractional** part of x

$\lfloor x \rfloor$ is called the **integer** part of x

By definition

$$0 \leq \{x\} < 1$$

and we write

$$x = \lfloor x \rfloor + \{x\}$$

Fractional Part of x

Fact 2

IF $x = n + \Theta,\ n \in Z$ and $0 \leq \Theta < 1$

THEN $n = \lfloor x \rfloor$ and $\Theta = \{x\}$

Proof

Let $x = n + \Theta,\ \Theta \in [0, 1)$. We get by **12.**

$\lfloor x \rfloor = \lfloor n + \Theta \rfloor = n + \lfloor \Theta \rfloor = n$ and

$$x = n + \Theta = \lfloor x \rfloor + \Theta = \lfloor x \rfloor + \{x\}$$

so $\Theta = \{x\}$

Properties

We have proved in **12.**

$$\lfloor x+n \rfloor = \lfloor x \rfloor + n \text{ for } x \in R,\ n \in Z$$

Question: What happens when we consider

$$\lfloor x+y \rfloor \quad \text{where } x \in R \text{ and } y \in R$$

Is it possible (and when it is possible) that for any $x, y \in R$

$$\lfloor x+y \rfloor = \lfloor x \rfloor + \lfloor y \rfloor$$

Properties

Consider

$$x = \lfloor x \rfloor + \{x\}, \quad \text{and} \quad y = \lfloor y \rfloor + \{y\}$$

We evaluate using **12.** $\lfloor x + n \rfloor = \lfloor x \rfloor + n$

$$\lfloor x + y \rfloor = \lfloor \lfloor x \rfloor + \lfloor y \rfloor + \{x\} + \{y\} \rfloor = \lfloor x \rfloor + \lfloor y \rfloor + \lfloor \{x\} + \{y\} \rfloor$$

By definition $0 \leq \{x\} < 1$ and $0 \leq \{y\} < 1$ so we have that

$$0 \leq \{x\} + \{y\} < 2$$

Hence we have proved the following property

Properties

Fact 3

For any $x, y \in R$

$$\lfloor x + y \rfloor = \lfloor x \rfloor + \lfloor y \rfloor \quad \text{when} \quad 0 \leq \{x\} + \{y\} < 1$$

and

$$\lfloor x + y \rfloor = \lfloor x \rfloor + \lfloor y \rfloor + 1 \quad \text{when} \quad 1 \leq \{x\} + \{y\} < 2$$

Examples

Example 1

Find $\lceil \log_2 35 \rceil$

Observe that $2^5 < 35 \leq 2^6$

Taking log with respect to base 2 , we get

$$5 < \log_2 35 \leq 6$$

We use property

10. $\lceil x \rceil = n$ if and only if $n-1 < x \leq n$

and get

$$\lceil \log_2 35 \rceil = 6$$

Examples

Example 2

Find $\lceil \log_2 32 \rceil$

Observe that $2^4 < 32 \leq 2^5$

Taking log with respect to base 2 , we get

$$4 < \log_2 32 \leq 5$$

We use property **10.** and get

$$\lceil \log_2 32 \rceil = 5$$

Examples

Example 3

Find $\lfloor \log_2 35 \rfloor$

Observe that $2^5 \leq 35 < 2^6$

Taking log with respect to base 2 , we get

$$5 \leq \log_2 32 < 6$$

We use property

$$\textbf{8.} \quad \lfloor\ x\ \rfloor = n \quad \text{if and only if} \quad n \leq x < n+1$$

and we get

$$\lfloor \log_2 32 \rfloor = 5 = \lceil \log_2 32 \rceil$$

Observation

Observe that 35 has 6 digits in its binary representation
$35 = (1000011)_2$ and $\lceil \log_2 35 \rceil = 6$

Question

Is the number of digits in binary representation of n always equal $\lceil \log_2 n \rceil$?

Answer: NO, it is not true

Consider $32 = (1000000)_2$

32 has 6 digits in its binary representation but

$$\lceil \log_2 32 \rceil = 5 \neq 6$$

Small Problem

Question: Can we develop a connection (formula) between $\lfloor \log_2 n \rfloor$ and number of digits (m) in the binary representation of n $(n > 0)$?

Answer: YES

Small Problem Solution

Let $n \neq 0,\ n \in N$ be such such that it has m bits in **binary representation**

Hence, by definition we have

$$n = a_{m-1}2^{m-1} + \ldots + a_0$$

and

$$2^{m-1} \leq n < 2^m$$

So we get **solution**

$$m - 1 \leq \log_2 n < m \quad \text{if and only if} \quad \lfloor \log_2 n \rfloor = m - 1$$

Small Fact and Exercise

We have proved the following

Fact 4

For any $n \neq 0, \; n \in N$ such such that it has m bits in **binary representation** we have that

$$\lfloor \log_2 n \rfloor = m - 1$$

Example

Take $n = 35, \; m = 6$ so $\lfloor \log_2 35 \rfloor = 6 - 1 = 5$

Take $n = 32, \; m = 6$ so we get $\lfloor \log_2 32 \rfloor = 6 - 1 = 5$

Exercise Develop similar formula for $\lceil \log_2 n \rceil$

Another Small Fact

Fact 5

For any $x \in R, \ x \geq 0$ the following property holds

$$\left\lfloor \sqrt{\lfloor x \rfloor} \right\rfloor = \lfloor \sqrt{x} \rfloor$$

Proof

Take $\left\lfloor \sqrt{\lfloor x \rfloor} \right\rfloor$

We proceed as follows

First we get rid of the outside $\lfloor \ \rfloor$ and **then** of the square root and of the inside $\lfloor \ \rfloor$

Proof

Let $m = \left\lfloor \sqrt{\lfloor x \rfloor} \right\rfloor$

By property

8. $\lfloor x \rfloor = n$ if and only if $n \leq x < n+1$

we get that

$$m = \left\lfloor \sqrt{\lfloor x \rfloor} \right\rfloor \text{ if and only if } m \leq \sqrt{\lfloor x \rfloor} < m+1$$

Squaring all sides of the inequality we get

$$(\star) \quad m = \left\lfloor \sqrt{\lfloor x \rfloor} \right\rfloor \text{ if and only if } m^2 \leq \lfloor x \rfloor < (m+1)^2$$

Proof

We proved that

$$(\star) \quad m = \left\lfloor \sqrt{\lfloor x \rfloor} \right\rfloor \quad \text{if and only if} \quad m^2 \leq \lfloor x \rfloor < (m+1)^2$$

Using property

$$\textbf{16.} \quad n \leq x \quad \text{if and only if} \quad n \leq \lfloor x \rfloor$$

on the left of inequality in $(\star)$ and property

$$\textbf{13.} \quad x < n \quad \text{if and only if} \quad \lfloor x \rfloor < n$$

on the right side of inequality in $(\star)$ we get

$$(\star\star) \quad m = \left\lfloor \sqrt{\lfloor x \rfloor} \right\rfloor \quad \text{if and only if} \quad m^2 \leq x < (m+1)^2$$

Proof

We already proved that

$$(\star\star) \quad m = \left\lfloor \sqrt{\lfloor x \rfloor} \right\rfloor \quad \text{if and only if} \quad m^2 \leq x < (m+1)^2$$

Now we retrace our steps backwards. First taking $\sqrt{x}$ on all sides of inequality $(\star\star)$ (all components are $\geq$0), we get

$$m = \left\lfloor \sqrt{\lfloor x \rfloor} \right\rfloor \quad \text{if and only if} \quad m \leq \sqrt{x} < m+1$$

We use now the property

$$\textbf{8.} \quad \lfloor x \rfloor = n \quad \text{if and only if} \quad n \leq x < n+1$$

and get

$$m = \left\lfloor \sqrt{\lfloor x \rfloor} \right\rfloor \quad \text{if and only if} \quad \lfloor \sqrt{x} \rfloor = m$$

and hence

$$\left\lfloor \sqrt{\lfloor x \rfloor} \right\rfloor = \lfloor \sqrt{x} \rfloor$$

It **ends** the **proof**

Exercise

Write a proof of

$$\left\lceil \sqrt{\lceil x \rceil} \right\rceil = \lceil \sqrt{x} \rceil$$

Question

How can we GENERALIZE our just proven properties for other functions then $f(x) = \sqrt{x}$?

For which functions $f = f(x)$ (class of which functions?) the following holds

$$\lfloor f(\lfloor x \rfloor) \rfloor = \lfloor f(x) \rfloor$$

and

$$\lceil f(\lceil x \rceil) \rceil = \lceil f(x) \rceil$$

Generalization

Here is a proper generalization of the **Fact 4**

Fact 5

Let $f: \quad R' \longrightarrow R$ where $R' \subseteq R$ is the domain of f

IF f = f(x) is continuous, monotonically increasing on its domain R' , and additionally has the following property **P**

$$\textbf{P} \qquad \text{if } f(x) \in Z \quad \text{then} \quad x \in Z$$

THEN for all $x \in R'$ for which the property **P** holds we have that

$$\lfloor f(\lfloor x \rfloor) \rfloor = \lfloor f(x) \rfloor$$

and

$$\lceil f(\lceil x \rceil) \rceil = \lceil f(x) \rceil$$

Fact 5 Proof

Proof

We want to show that under assumption that f is continuous, monotonic, increasing on its domain R' the property

$$\lceil f(\lceil x \rceil) \rceil = \lceil f(x) \rceil$$

holds for all $x \in R'$ for which the property **P** holds

Case 1 take $x = \lceil x \rceil$

We get

$$\lceil f(x) \rceil = \lceil f(\lceil x \rceil) \rceil$$

is trivial as in this case we have that $x \in Z$

Fact 5 Proof

Case 2 take $x \neq \lceil x \rceil$

By definition $x < \lceil x \rceil$ and function f is monotonically increasing so we have

$$f(x) < f(\lceil x \rceil)$$

By the fact that $\lceil \ \rceil$ is non- decreasing , i.e.

If $x < y$ then $\lceil x \rceil \leq \lceil y \rceil$

we get

$$\lceil f(x) \rceil \leq \lceil f(\lceil x \rceil) \rceil$$

Now we show that $<$ is impossible

Hence we will have $=$

Fact 5 Proof

Assume

$$\lceil f(x) \rceil < \lceil\, f(\lceil x \rceil)\, \rceil$$

Since f is continuous, then there is y , such that

$$f(y) = \lceil f(x) \rceil$$

and

$$(\star) \quad x \leq y < \lceil x \rceil$$

But $f(y) = \lceil f(x) \rceil$, i.e. $f(y) \in Z$ hence by property **P** we get

$$(\star\star) \quad y \in Z$$

Observe that $(\star)$ and $(\star\star)$ are **contradictory** as there is no $y \in Z$ between x and $\lceil x \rceil$ and this **ends the proof**

Exercises

Exercise 1

Prove the first part of the **Fact 5**, i.e.

$$\left\lfloor \sqrt{\lfloor f(x) \rfloor} \right\rfloor = \left\lfloor \sqrt{f(x)} \right\rfloor$$

Exercise 2

Prove that for any $x \in R, \quad n, m \in Z$

$$1. \quad \left\lfloor \frac{x+m}{n} \right\rfloor = \left\lfloor \frac{\lfloor x \rfloor + m}{n} \right\rfloor$$

and

$$2. \quad \left\lceil \frac{x+m}{n} \right\rceil = \left\lceil \frac{\lceil x \rceil + m}{n} \right\rceil$$

Exercise 2 Solution

Let's prove

$$1. \quad \left\lfloor \frac{x+m}{n} \right\rfloor = \left\lfloor \frac{\lfloor x \rfloor + m}{n} \right\rfloor$$

Proof for $\lceil \ \rceil$ is carried similarly and is left as an exercise

Take a function

$$f(x) = \frac{x+m}{n}$$

for $n, m \in Z, \ x \in R$

Observe that

$$f(x) = \frac{x+m}{n} = \frac{x}{n} + \frac{m}{n}$$

is a line $f(x) = ax + b$ and hence is continuous, monotonically increasing

Exercise 2 Solution

We have to check now if the property **P**

$$\textbf{P} \qquad \text{if } f(x) \in Z \text{ then } x \in Z$$

holds for it, i.e. to check if all assumptions of the **Fact 5** are fulfilled

Then by the **Fact 5** we will get that

$$\lfloor f(\lfloor x \rfloor) \rfloor = \lfloor f(x) \rfloor$$

i.e.

$$\left\lfloor \frac{\lfloor x \rfloor + m}{n} \right\rfloor = \left\lfloor \frac{x + m}{n} \right\rfloor$$

Exercise 2 Solution

Poof that the property **P** holds for

$$f(x) = \frac{x+m}{n}$$

Assume $f(x) \in Z$, i.e. there is $k \in Z$ such that

$$\frac{x+m}{n} = k$$

It means that

$$x + m = nk$$

and

$$x = nk - m \in Z \quad \text{as} \quad n, k, m \in Z$$

Integers in the Intervals

Intervals

Standard Notation and definition of a Closed Interval

$$[\alpha,\ \beta] = \{x \in R : \quad \alpha \le x \le \beta\}$$

Book Notation

$$[\alpha \ldots \beta] = \{x \in R : \quad \alpha \le x \le \beta\}$$

We use book notation, because $[P(x)]$ denotes in the book the characteristic function of P(x)

Intervals

Closed Interval

$$[\alpha,\ \beta] = \{x \in R : \quad \alpha \le x \le \beta\} \ = [\alpha \ldots \beta]$$

Open Interval

$$(\alpha,\ \beta) = \{x \in R : \quad \alpha < x < \beta\} \ = (\alpha \ldots \beta)$$

Half Open Interval

$$[\alpha,\ \beta) = \{x \in R : \quad \alpha \le x < \beta\} \ = [\alpha \ldots \beta)$$

Half Open Interval

$$(\alpha,\ \beta] = \{x \in R : \quad \alpha < x \le \beta\} \ = (\alpha \ldots \beta]$$

Integers in the Intervals

Problem

How many integers are there in the intervals?

In other words, for

$A = \{ x \in Z : \alpha \leq x \leq \beta \}$

$A = \{ x \in Z : \alpha < x \leq \beta \}$

$A = \{ x \in Z : \alpha \leq x < \beta \}$

$A = \{ x \in Z : \alpha < x < \beta \}$

We want to find $|A|$

Solution

We bring our $\lceil\rceil$, $\lfloor\rfloor$ properties **13. - 16.**

13. $x < n$ if and only if $\lfloor x \rfloor < n$

14. $n < x$ if and only if $n < \lceil x \rceil$

15. $x \leq n$ if and only if $\lceil x \rceil \leq n$

16. $n \leq x$ if and only if $n \leq \lfloor x \rfloor$

and we get for $\alpha, \beta \in R$ and $n \in Z$

$$\alpha \leq n < \beta \quad \text{if and only if} \quad \lceil \alpha \rceil \leq n < \lceil \beta \rceil$$

$$\alpha < n \leq \beta \quad \text{if and only if} \quad \lfloor \alpha \rfloor \leq n < \lfloor \beta \rfloor$$

Integers in the Intervals

Solution

$[\alpha...\beta)$ contains exactly $\lceil\beta\rceil - \lceil\alpha\rceil$ integers

$(\alpha...\beta]$ contains exactly $\lfloor\beta\rfloor - \lfloor\alpha\rfloor$ integers

$[\alpha...\beta]$ contains exactly $\lfloor\beta\rfloor - \lceil\alpha\rceil + 1$ integers

We must assume $\alpha \neq \beta$ to evaluate

$(\alpha...\beta)$ contains exactly $\lceil\beta\rceil - \lfloor\alpha\rfloor - 1$ integers

We

because $(\alpha...\alpha) = \emptyset$ and can't contain -1 integers

Integers in the Intervals

INTERVAL	Number of INTEGERS	RESTRICTIONS
$[\alpha ... \beta]$	$\lfloor \beta \rfloor$-$\lceil \alpha \rceil + 1$	$\alpha \leq \beta$
$[\alpha ... \beta)$	$\lceil \beta \rceil$-$\lceil \alpha \rceil$	$\alpha \leq \beta$
$(\alpha ... \beta]$	$\lfloor \beta \rfloor$-$\lfloor \alpha \rfloor$	$\alpha \leq \beta$
$(\alpha ... \beta)$	$\lceil \beta \rceil$-$\lfloor \alpha \rfloor - 1$	$\alpha < \beta$

Casino Problem

Casino Problem

There is a roulette wheel with 1,000 slots numbered $1 \ldots 1,000$

IF the number n that comes up on a spin is divisible by $\lfloor \sqrt[3]{n} \rfloor$ what we write as

$$\lfloor \sqrt[3]{n} \rfloor \mid n$$

THEN n is the **winner**

Reminder

We **define divisibility** $\mid$ in a standard way:

$k \mid n$ if and only if there exists $m \in Z$ such that $n = km$

Average Winnings

In the game Casino pays \$5 if you are the **winner**; but the **loser** has to pay \$1

Can we expect to make money if we play this game?

Let's compute average winnings, i.e. the amount we win (or lose) per play

Denote

W - number of **winners**

L - number of **losers** and L = 1000 - W

Strong Rule: each number comes once during 1000 plays

Casino Winnings

Under the **Strong Rule** we win 5W and lose L dollars and the average winnings in 1000 plays is

$$\frac{5W - L}{1000} = \frac{5W - (1000 - W)}{1000} = \frac{6W - 1000}{1000}$$

We have **advantage** if

$$6W > 1000$$

i.e. when

$$W > 167$$

Casino Winnings

Answer

IF there is 167 or more winners and we play under the

Strong Rule: each number comes once during 1000 plays

THEN we have the **advantage**, otherwise **Casino wins**

Number of Winners

Problem

How to **count** the number of winners among 1 to 1000

Method

Use summation

$$W = \sum_{n=1}^{1000} [\text{n is a winner}]$$

Casino Problem

Reminder of Casino Problem

There is a roulette wheel with 1,000 slots numbered $1 \ldots 1,000$

IF the number n that comes up on a spin is divisible by $\lfloor \sqrt[3]{n} \rfloor$, i.e. $\sqrt[3]{n} \rfloor \mid n$

THEN n is the **winner**

The summations becomes

$$W = \sum_{n=1}^{1000} [\text{n is a winner}] = \sum_{n=1}^{1000} [\lfloor \sqrt[3]{n} \rfloor \mid n]$$

where we **define divisibility** $\mid$ in a standard way

$k \mid n$ if and only if there exists $m \in Z$ such that $n = km$

Book Solution

Here are 7 steps of our BOOK solution

1. $W = \sum_{n=1}^{1000} [n \textit{ is a winner}\,] = \sum_{n=1}^{1000} \left[\lfloor \sqrt[3]{n} \rfloor \mid n\right]$
2. $W = \sum_{k,n} \left[k = \lfloor \sqrt[3]{n} \rfloor\right] [k|n] [1 \le n \le 1000]$
3. $W = \sum_{k,n,m} \left[k^3 \le n < (k+1)^3\right] [n = km] [1 \le n \le 1000]$
4. $W = 1 + \sum_{k,m} \left[k^3 \le km < (k+1)^3\right] [1 \le k < 10]$
5. $W = 1 + \sum_{k,m} \left[m \in \left[k^2 \ldots \frac{(k+1)^3}{k}\right)\right] [1 \le k < 10]$
6. $W = 1 + \sum_{1 \le k < 10} \left(\lceil k^2 + 3k + 3 + \frac{1}{k} \rceil - \lceil k^2 \rceil\right)$
7. $W = 1 + \sum_{1 \le k < 10} (3k + 4) = 1 + \frac{7+31}{2} 9 = 172$

Class Problem

Here are the BOOK comments

1. This derivation merits careful study

2. The only "difficult" maneuver is the decision between lines **3** and **4** to treat n =1000 as a special case

3. The inequality $k^3 \leq n < (k+1)^3$ does not combine easily with $1 \leq n \leq 1000$ when k=10

Book Solution Comments

Class Problem

Write down explanation of **each step** with **detailed** justifications (Facts, definitions) why they are correct

By doing so fill all gaps in the **proof** that

$$W = \sum_{n=1}^{1000} [\lfloor \sqrt[3]{n} \rfloor \mid n] = 172$$

This problem can also appear on your tests

QUESTIONS about Book Solution

Here are **questions** to answer about the steps in the BOOK solution

1 W = $\sum_{n=1}^{1000} [n \text{ is a winner}] = \sum_{n=1}^{1000} \left[\lfloor \sqrt[3]{n} \rfloor \mid n \right]$

Q1 Explain why $[n \text{ is a winner}] = \left[\lfloor \sqrt[3]{n} \rfloor \mid n \right]$

2 W = $\sum_{k,n} \left[k = \lfloor \sqrt[3]{n} \rfloor \right] [k|n] [1 \leq n \leq 1000]$

Q2 Explain why and how we have changed a sum $\sum_{n=1}^{1000}$ into a sum $\sum_{k,n}$ and

$$\sum_{n=1}^{1000} \left[\lfloor \sqrt[3]{n} \rfloor \mid n \right] = \sum_{k,n} \left[k = \lfloor \sqrt[3]{n} \rfloor \right] [k|n] [1 \leq n \leq 1000]$$

QUESTIONS about Book Solution

3 $W = \sum_{k,n,m} \left[k^3 \le n < (k+1)^3\right] [n = km] [1 \le n \le 1000]$

Q3 Explain why

$$\left[k = \lfloor \sqrt[3]{n} \rfloor\right] [k \mid n] = \left[k^3 \le n < (k+1)^3\right] [n = km]$$

Explain why and how we have changed sum $\sum_{k,n}$ into a sum $\sum_{k,n,m}$

QUESTIONS about Book Solution

4 W = 1 + $\sum_{k,m} \left[k^3 \leq km < (k+1)^3\right] [1 \leq k < 10]$

Q4 There are three sub- questions; the last one is one of the book questions

1. Explain why

$$\left[k^3 \leq n < (k+1)^3\right] [n = km] [1 \leq n \leq 1000] =$$
$$\left[k^3 \leq km < (k+1)^3\right] [1 \leq k < 10]$$

2. Explain why and how we have changed sum $\sum_{k,n,m}$ into
a sum $\sum_{k,m}$

3. Explain HOW and why we have got $1 + \sum_{k,m}$

QUESTIONS about Book Solution

5 $$W = 1 + \sum_{k,m} \left[m \in \left[k^2 \dots \frac{(k+1)^3}{k} \right) \right] [1 \leq k < 10]$$

Q5 Explain transition

$$\left[k^3 \leq km < (k+1)^3 \right] = \left[m \in \left[k^2 \dots \frac{(k+1)^3}{k} \right) \right]$$

QUESTIONS about Book Solution

6 W = 1 + $\sum_{1 \le k < 10} \left(\lceil k^2 + 3k + 3 + \frac{1}{k} \rceil - \lceil k^2 \rceil \right)$

Q6 Explain (prove) why

$$\sum_{k,m} \left[m \in \left[k^2 \ldots \frac{(k+1)^3}{k} \right) \right] [1 \le k < 10] =$$

$$\sum_{1 \le k < 10} \left(\lceil k^2 + 3k + 3 + \frac{1}{k} \rceil - \lceil k^2 \rceil \right)$$

Observe that $\left[m \in \left[k^2 \ldots \frac{(k+1)^3}{k} \right) \right]$ is a characteristic function and $\left(\lceil k^2 + 3k + 3 + \frac{1}{k} \rceil - \lceil k^2 \rceil \right)$ is an integer

QUESTIONS about Book Solution

7 W = 1 + $\sum_{1 \le k < 10} (3k+4) = 1 + \frac{7+31}{2} 9 = 172$

Q7 Explain (prove) why

$$\left(\lceil k^2 + 3k + 3 + \tfrac{1}{k} \rceil - \lceil k^2 \rceil\right) = (3k+4)$$

Before we giving answers to Q1 - Q7 we need to review some of the SUMS material