cse547, math547
DISCRETE MATHEMATICS
Lectures Content Final
Infinite Series, Chapter 3 and Chapter 4

Professor Anita Wasilewska

Spring 2014

CHAPTER 2
PART 5: INFINITE SUMS (SERIES)

Here are Definitions, Basic Theorems and Examples you must know

Series
Definitions, Theorems, Simple Examples

Must Know STATEMENTS- do not need to PROVE the Theorems

Definition

If the limit $\lim_{n\to\infty} S_n$ **exists** and **is finite**, i.e.

$$\lim_{n\to\infty} S_n = S,$$

then we say that the infinite sum $\Sigma_{n=1}^{\infty} a_n$ **converges** to S and we write

$$\Sigma_{n=1}^{\infty} a_n = \lim_{n\to\infty} \Sigma_{k=1}^{n} a_k = S,$$

otherwise the infinite sum **diverges**

Definitions, Theorems, Simple Examples

Show

The infinite sum $\Sigma_{n=1}^{\infty} (-1)^n$ **diverges**

The infinite sum $\Sigma_{n=0}^{\infty} \frac{1}{(k+1)(k+2)}$ **converges** to 1

Definitions, Theorems, Simple Examples

Theorem 1

If the infinite sum

$$\Sigma_{n=1}^{\infty} a_n \quad \text{converges}, \quad \text{then} \lim_{n\to\infty} a_n = 0$$

Definition 5

An infinite sum

$$\Sigma_{n=1}^{\infty} (-1)^{n+1} a_n, \quad \text{for} \quad a_n \geq 0$$

is called **alternating infinite sum** (alternating series)

Theorem 6 Comparing the series

Let $\Sigma_{n=1}^{\infty} a_n$ be an infinite sum and $\{b_n\}$ be a sequence such that

$$0 \leq b_n \leq a_n \quad \text{for all n}$$

If the infinite sum $\Sigma_{n=1}^{\infty} a_n$ **converges**

then $\Sigma_{n=1}^{\infty} b_n$ also **converges** and

$$\Sigma_{n=1}^{\infty} b_n \leq \Sigma_{n=1}^{\infty} a_n$$

Use **Theorem 6** to prove that the series,

$$\Sigma_{n=1}^{\infty} \frac{1}{(n+1)^2}$$

converges

Theorem 7 (D'Alambert's Criterium)

If $a_n \geq 0$ and $\lim\limits_{n\to\infty} \dfrac{a_{n+1}}{a_n} < 1$

then the series $\sum\limits_{n=1}^{\infty} a_n$ **converges**

Theorem 8 (Cauchy's Criterium)

If $a_n \geq 0$ and $\lim\limits_{n\to\infty} \sqrt[n]{a_n} < 1$

then the series $\sum\limits_{n=1}^{\infty} a_n$ **converges**

Definitions, Theorems, Simple Examples

Theorem 9 (Divergence Criteria)

If $a_n \geq 0$ and $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} > 1$ or $\lim_{n\to\infty} \sqrt[n]{a_n} > 1$

then the series $\sum_{n=1}^{\infty} a_n$ **diverges**

Prove

The series $\sum_{n=1}^{\infty} \frac{1}{(n+1)^2}$ does not react on D'Alambert's Criterium (Theorem 7)

Definitions, Theorems, Simple Examples

STUDY ALL EXAMPLES from Lecture 10

CHAPTER 3
INTEGER FUNCTIONS

Here is the **proofs** in course material you need to know for **Midterm 2** and **Final**

Plus the regular Homeworks Problems

PART1: Floors and Ceilings

Prove the following

Fact 3

For any $x, y \in R$

$$\lfloor x+y \rfloor = \lfloor x \rfloor + \lfloor y \rfloor \quad \text{when} \quad 0 \leq \{x\} + \{y\} < 1$$

and

$$\lfloor x+y \rfloor = \lfloor x \rfloor + \lfloor y \rfloor + 1 \quad \text{when} \quad 1 \leq \{x\} + \{y\} < 2$$

Fact 5

For any $x \in R,\ x \geq 0$ the following property holds

$$\left\lfloor \sqrt{\lfloor x \rfloor} \right\rfloor = \lfloor \sqrt{x} \rfloor$$

PART1: Floors and Ceilings

Prove the following properties of characteristic functions

F1 For any predicates P(k), Q(k)

$$[P(k) \cap Q(k)] = [P(k)][Q(k)]$$

F2 For any predicates P(k), Q(k)

$$[P(k) \cup Q(k)] = [P(k)] + [Q(k)] - [P(k) \cap Q(k)]$$

Prove the Combined Domains Property

Property 4

$$\sum_{Q(k)\cup R(k)} a_k = \sum_{Q(k)} a_k + \sum_{R(k)} a_k - \sum_{Q(k)\cap R(k)} a_k$$

where, as before,

$k \in K$ and $K = K_1 \times K_2 \cdots \times K_i$ for $1 \leq i \leq n$

and the above formula represents single (i =1) and multiple ($i > 1$) sums

PART1: Floors and Ceilings

Study all 7 steps of our explanations to BOOK solution
I will give you **ONE to write in full** on the test

1 $W = \sum_{n=1}^{1000} [n \text{ is a winner}] = \sum_{n=1}^{1000} [\lfloor \sqrt[3]{n} \rfloor \mid n]$

2 $W = \sum_{k,n} \left[k = \lfloor \sqrt[3]{n} \rfloor\right] [k|n] [1 \leq n \leq 1000]$

3 $W = \sum_{k,n,m} \left[k^3 \leq n < (k+1)^3\right] [n = km] [1 \leq n \leq 1000]$

4 $W = 1 + \sum_{k,m} \left[k^3 \leq km < (k+1)^3\right] [1 \leq k < 10]$

5 $W = 1 + \sum_{k,m} \left[m \in \left[k^2 \ldots \frac{(k+1)^3}{k}\right)\right] [1 \leq k < 10]$

6 $W = 1 + \sum_{1 \leq k < 10} \left(\lceil k^2 + 3k + 3 + \frac{1}{k} \rceil - \lceil k^2 \rceil\right)$

7 $W = 1 + \sum_{1 \leq k < 10} (3k+4) = 1 + \frac{7+31}{2} 9 = 172$

PART2: Spectrum Partitions

Prove the following properties

$$\textbf{P1} \qquad \sum_{k} [R(k)] = \sum_{R(k)} 1 = | R(k) |$$

$$\textbf{P2} \quad \sum_{k,m} [P(m)]\,[Q(k)] = \sum_{Q(k)} \sum_{P(m)} 1 = \sum_{Q(k)} | P(m) |$$

where we denote for short

$$| P(m) | = | \{ m \in N : P(m) \} |$$

Justify that

$$N(\alpha, n) = \sum_{k>0} \left[k < \frac{n+1}{\alpha} \right]$$

Write a detailed proof of

$$N(\alpha, n) = \left\lceil \frac{n+1}{\alpha} \right\rceil - 1$$

Write a detailed proof of

PART2: Spectrum Partitions

Prove the following

Fact P2

If $|A| + |B| = |X|$ and $A \neq \emptyset$, $B \neq \emptyset$ and $A \cap B = \emptyset$

then the sets A, B form a **finite partition** of X

Spectrum Fact

$$Spec(\sqrt{2}) \cap Spec(2 + \sqrt{2}) = \emptyset$$

Finite Spectrum Partition Theorem

1. $A_n \neq \emptyset$ and $B_n \neq \emptyset$

2. $A_n \cap B_n = \emptyset$

3. $A_n \cup B_n = \{1, 2, \ldots n\}$

PART2: Spectrum Partitions

Prove - use your favorite proof out of the two I have provided

Spectrum Partition Theorem

1. $Spec(\sqrt{2}) \neq \emptyset$ and $Spec(2+\sqrt{2}) \neq \emptyset$
2. $Spec(\sqrt{2}) \cap Spec(2+\sqrt{2}) = \emptyset$
3. $Spec(\sqrt{2}) \cup Spec(2+\sqrt{2}) = N - \{0\}$

PART3: Sums

Write detailed evaluation of

$$\sum_{0 \le k < n} \lfloor \sqrt{k} \rfloor$$

Hint: use

$$\sum_{0 \le k < n} \lfloor \sqrt{k} \rfloor = \sum_{0 \le k < n} \quad \sum_{m \ge 0,\ m = \lfloor \sqrt{k} \rfloor} m$$

Chapter 4 Material in the Lecture 12

JUSTIFY correctness of the following example and be ready to do similar problems upwards or downwards

Represent 19151 in a system with base 12

Example

$$19151 = 1595 \cdot 12 + 11$$

$$1595 = 132 \cdot 12 + 11$$

$$132 = 11 \cdot 12 + 0$$

$$a_0 = 11, \quad a_1 = 11, \quad a_2 = 0, \quad a_3 = 11$$

So we get

$$19151 = (11, 0, 11, 11)_{12}$$

Write a proof of **Step 1** or **Step 2** of the Proof of the Correctness of Euclid Algorithm

You can use Lecture OR BOOK formalization and proofs

Use Euclid Algorithms to prove

When a product ac of two natural numbers is divisible by a number b that is **relatively prime** to a, the factor c must be divisible by b

Use Euclid Algorithms to prove the following

Fact

$$gcd(ka, kb) = k \cdot gcd(a, b)$$

Theorems, Proofs and Problems

Prove:

Any common multiple of a and b is **divisible** by lcm(a,b)

Prove the following

$$\forall_{p,q_1 q_2 \ldots q_n \in P}\ (p \mid \prod_{k=1}^{n} q_k \ \Rightarrow\ \exists_{1 \leq i \leq n}\ (p = q_i\ )\ )$$

Write down a formal formulation (in all details) of the **Main Factorization Theorem** and its General Form

Theorems, Proofs and Problems

Prove that the representation given by Main Factorization Theorem is unique

Explain why and show that $18 =< 1, 2 >$

Prove

$$k = gcd(m, n) \quad \text{if and only if} \quad k_p = min\{m_p, n_p\}$$

$$k = lcd(m, n) \quad \text{if and only if} \quad k_p = max\{m_p, n_p\}$$

Let

$$m = 2^0 \cdot 3^3 \cdot 5^2 \cdot 7^0 \qquad n = 2^0 \cdot 3^1 \cdot 5^0 \cdot 7^3$$

Evaluate gcd(m, n) and k = lcd(m, n)

Exercises

1. Use Facts 6-8 to prove

Theorem 5

For any $a, b \in Z^{+}$ such that lcm(a,b) and gcd(a, b) exist

$$lcm(a, b) \cdot gcd(a, b) = ab$$

2. Use **Theorem 5** and the BOOK version of Euclid Algorithm to express lcm(n mod m, m) when $nmodm \neq 0$

This is Ch4 Problem 2