CSE541 Take Home MIDTERM 2 due April 26 in class
Spring 2011
100pts

NAME **ID:**

Each QUESTION except EXTRA Credit Questions is **20pts**

PART ONE

Remark: read Lecture Notes 1 for intuitive introduction to Predicate Logic Languages. You don't need a more formal definition or any extra material to solve the problems below. All what is needed is defined in the definitions provided below.

Definitions

Let $\mathcal{L}$ denote a language of classical logic, propositional, or predicate , with full set of propositional connectives with the set $\mathcal{F}$ of formulas.

Definition 1 For any $A \in \mathcal{F}$ we write $M \models A$ to denote that M is **a model for** A.

If $\mathcal{L}$ is **a propositional language**, M is called a **propositional model**, if $\mathcal{L}$ is a **predicate (logic) language,** M is called a **predicate model.**

Example 1 Let A be a propositional formula $((a \cup \neg b) \Rightarrow b)$. A model M is any truth assignment $v : VAR \longrightarrow \{T, F\}$, such that $v(a) = F, v(b) = T$. Observe that the v is not the only model for A.

Example 2 Let A be a predicate logic formula formula $\exists x P(x, c)$. where P is a two arguments predicate symbol representing any two argument relation, c is a symbol for a constant. $P(x, c)$ is an atomic formula. **The structure** $M = (N, >, 0)$ is **a model** for A because when we interpret predicate symbol P as $>$ and a constant symbol c as $0 \in N$ we obtain a true statement $\exists x(x > 0)$ about natural numbers.

The same structure M is not a model for a formula $\forall x P(x, c)$, as $\forall x(x > 0)$ is a false statement about natural numbers.

Observe that under classical semantics the structure $M = (N, >, 0)$ is a model for a formula $(\forall x P(x, c) \Rightarrow \exists x P(x, c))$, but not for a formula $(\exists x P(x, c) \Rightarrow \forall x P(x, c))$.

Propositional Structure M is any truth assignment $v : VAR \longrightarrow \{T, F\}$.

Formal Definition of $M \models A$ for propositional structure $M = v$ is exactly what we have defined from the beginning of the course i.e.

$$v \models A \text{ iff } \quad v^*(A) = T.$$

Predicate Structure M is a structure $M = (U, R_1, ..R_k, x_1, ...x_n)$, where $X \neq \emptyset$ is called a UNIVERSE of the structure, $R_1, ..R_k$ are certain RELATIONS defined on X (they correspond to the predicates in your formulas) and $x_1, ...x_n$ special elements of the Universe that correspond to to constants from the language.

Remark The relationship predicate formula - structure is the "inverse translation" to the one between mathematical statements and logic formulas explained in Lecture Notes 1.

Informal Definition of $M \models A$ for predicate structure M.

Structure $M = (U, R_1, ..R_k, x_1, ...x_n)$ is **a model** for a formula A iff the translation of A into a concrete statement about the Universe U of the structure M is **a TRUE statement** about this Universe.

Definition 2 A formula $A \in F$ is a classical tautology, what we write $\models A$ if and only if all structures M are models for A, i.e. $M \models A$, for all M.

Definition 3 M is a **model for a set** (finite or infinite) $\mathcal{G} \subseteq \mathcal{F}$ **of formulas** of $\mathcal{L}$ if and only if $M \models B$ for all $B \in \mathcal{G}$.

We denote it by $M \models \mathcal{G}$.

Definition 4 A set $\mathcal{G}$ of formulas is called **consistent** if and only if **it has a model**, i.e. there is M, such that $M \models \mathcal{G}$.

Otherwise $\mathcal{G}$ is called **inconsistent.**

Remark Definition 4 provides a **SEMANTIC** notion of consistency/inconsistency. In the proof 2 of the Completeness Theorem we have introduced and used a SYNTACTIC notion, i.e. we used the notion of a proof to define it.

Definition 5 A formula A is called **independent** from a set of formulas $\mathcal{G}$ if and only of there are M_1, M_2 such that

$$M_1 \models \mathcal{G} \cup \{A\} \text{ and } \quad M_2 \models \mathcal{G} \cup \{\neg A\},$$

i.e. when **when both** $\mathcal{G} \cup \{A\}$ **and** $\mathcal{G} \cup \{\neg A\}$ **are consistent.**

Definition 6 We say that the set $\mathcal{G}$ **semantically entails** a formula A if and only if for any M,

$$M \models \mathcal{G} \text{ implies that } M \models A.$$

We denote it by $\mathcal{G} \models A$.

QUESTION 1

(a) Given a set

$$S = \{((a \cap b) \Rightarrow b),\ (a \cup b), \neg a\}.$$

1. Show that S is **consistent**.

2. Show that a formula $A = (\neg a \cap b)$ is **not independent** of S.

3. Find an infinite number of formulas that are independent of S.

4. Give an example of **an infinite consistent** set S (propositional language).

QUESTION 2 (**10 extra credit**)

Given a set S of formulas:

$$S = \{\forall x((R(x,y) \cap R(y,z)) \Rightarrow R(x,z)),\ \forall x R(x,x)\}.$$

Remember: $R(x,y)$ is a two argument predicate representing a binary relation.

1. Show that S is **consistent**.

2. Show that a formula $A = \forall x(R(x,y) \Rightarrow R(y,x))$ is **independent** of S.

QUESTION 3

1. Show that if $S = \emptyset$, then for any formula A of $\mathcal{F}$ of propositional or predicate language,

$$S \models A \text{ implies that } \models A.$$

2. Show that there is $S \neq \emptyset$, such that for any A, such that $\models A$, $S \models A$.

3. Show that if $S \subseteq \mathcal{F}$ is **inconsistent** then $\{A : S \models A\} = \mathcal{F}$.

PART 2

QUESTION 4

Consider a system **RS1** obtained from **RS** by changing the sequence Γ' into Γ in all of the rules of inference of **RS**.

1. Explain why the system **RS1** is sound. You can use the Soundness of the system **RS.**

2. Construct **TWO** decomposition trees of

$$(\neg(\neg a \Rightarrow (a \cap \neg b)) \Rightarrow (\neg a \cap (\neg a \cup \neg b)))$$

2. If there is a tree constructed that is not a proof, construct the counter-model determined by that tree. Justify that it is a counter-model.

QUESTION 5

1. Define shortly, in your own words, for any A, the decomposition tree $\mathbf{T}_A$ in **RS1** as defined in QUESTION 3. Justify why your definition is correct. Show that in **RS1** decomposition tree may not be unique.

2. Prove the **Completeness Theorem** for **RS1** (do not need to prove soundness).

QUESTION 6 (EXTRA 10pts) Write a procedure $TREE_A$ that for any formula A of **RS1** it produces its UNIQUE decomposition tree and prove **COMPLETENESS** of this procedure.

QUESTION 7

1. Let **GL** be the Gentzen style proof system defined in chapter 11.

1. Prove, by constructing a proper decomposition trees that

$$\nvdash_{\mathbf{GL}}((a \Rightarrow b) \Rightarrow (\neg b \Rightarrow a)).$$

2 Use the above to prove, without use of the Completeness Theorem that

$$\not\models \ ((a \Rightarrow b) \Rightarrow (\neg b \Rightarrow a)).$$