QUESTION 1 (10pts)

1. Use Dedekind theorem to prove that the set \( R \) of real numbers is infinite.

2. Find a function \( f \) that is 1−1 and maps \( R \) \( \text{onto} \) \( R - \{1, 8, 10\} \).

QUESTION 2 (20pts)

Here are some definitions; some of them are known to you and put as a reminder.

Definition 1 By a \textbf{m-valued semantics} \( S_m \) for a propositional language \( L = \{\neg, \land, \lor, \Rightarrow\} \) we understand any definition of of connectives \( \neg, \land, \lor, \Rightarrow \) as operations on a set \( V_m = \{ v_1, v_2, \ldots v_m \} \) of logical values.

We assume that \( v_1 \leq v_2 \leq \ldots \leq v_m \), i.e. \( V_m \) is totally ordered by a certain relation \( \leq \) with \( v_1, v_m \) being smallest and greatest elements, respectively. We denote \( v_1 = F, v_m = T \) and call them (total) False and Truth, respectively.
Definition 2 Let \( \text{VAR} \) be a set of propositional variables of \( \mathcal{L} \) and let \( S_m \) be any \( m \)-valued semantics for \( \mathcal{L} \). A truth assignment \( v : \text{VAR} \rightarrow V_m \) is called a \( S_m \) model for a formula \( A \) of \( \mathcal{L} \) iff \( v(A) = T \) and logical value \( v(A) \) is evaluated accordingly to the semantics \( S_m \). We denote it symbolically as
\[
v \models_{S_m} A.
\]
Any \( v \) such that \( v \) is not a \( S_m \) model for a formula \( A \) is called a counter-model for \( A \).

Definition 3 A formula \( A \) of \( \mathcal{L} \) is called a \( S_m \) tautology iff \( v \models_{S_m} A \), for all \( v \). We denote it by \( \models_{S_m} A \), and \( \models A \) for classical semantics tautologies.

Definition 4 A proof system \( S \) is complete with respect to a semantics \( S_m \) iff for any formula \( A \), the following holds:
\( A \) is provable in \( S \) iff \( A \) is \( S_m \) tautology.

Q2 Part one \( (15 \text{pts}) \)
Let \( S_3 \) be a 3-valued semantics for \( \mathcal{L}_{\{\neg, \cap, \cup, \Rightarrow\}} \) defined as follows.
\( V_3 = \{F, U, T\} \), for \( F \leq U \leq T \) and

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\( a \cup b = \min\{a, b\} \), \( a \Rightarrow b = \neg a \cup b \), for any \( a, b \in V_3 \).

Consider the following classical tautologies:

\( A_1 = (A \cup \neg A), \quad A_2 = (A \Rightarrow (B \Rightarrow A)). \)

(a) Find \( S_3 \) counter-models for \( A_1, A_2 \), if exist. Use shorthand notation.
(b) Define a 2-valued semantics $S_2$ for $L$, such that none of $A_1, A_2$ is a $S_2$ tautology. Verify your results. Use shorthand notation.

(c) Define a 3-valued semantics $C_3$ for $L$, such that both $A_1, A_2$ are a $C_3$ tautologies. Verify your results. Use shorthand notation.

Q2 Part Two (5pts)

Let $S = (L, A_1, A_2, A_3, MP)$ be a proof system with axioms:

A1   $(A \Rightarrow (B \Rightarrow A))$,
A2   $((A \Rightarrow (B \Rightarrow C)) \Rightarrow ((A \Rightarrow B) \Rightarrow (A \Rightarrow C)))$,
A3   $((\neg B \Rightarrow \neg A) \Rightarrow ((\neg B \Rightarrow A) \Rightarrow B))$,

The system $S$ is complete with respect to classical semantics.

Verify whether $S$ is complete with respect to 3-valued semantics $S_3$, as defined at the beginning of this question.
QUESTION 3 (15pts)

Let $S$ be from QUESTION 2, PART 2.

The following Lemma holds in the system $S$.

**LEMMA** For any $A, B, C \in \mathcal{F}$,

(a) $(A \Rightarrow B), (B \Rightarrow C) \vdash_H (A \Rightarrow C),$

(b) $(A \Rightarrow (B \Rightarrow C)) \vdash_H (B \Rightarrow (A \Rightarrow C)).$

Complete the proof sequence (in $S$)

$B_1, ..., B_9$

by providing comments how each step of the proof was obtained.

$B_1 = (A \Rightarrow B)$

$B_2 = (\neg\neg A \Rightarrow A)$

Already PROVED

$B_3 = (\neg\neg A \Rightarrow B)$

$B_4 = (B \Rightarrow \neg\neg B)$

Already PROVED

$B_5 = (\neg\neg A \Rightarrow \neg\neg B)$

$B_6 = ((\neg\neg A \Rightarrow \neg\neg B) \Rightarrow (\neg B \Rightarrow \neg A))$

Already PROVED

$B_7 = (\neg B \Rightarrow \neg A)$

$B_8 = (A \Rightarrow B) \vdash (\neg B \Rightarrow \neg A)$

$B_9 = ((A \Rightarrow B) \Rightarrow (\neg B \Rightarrow \neg A))$
QUESTION 4 (35pts)
Consider any proof system \( S \),
\[
S = (\mathcal{L}_{\cap, \cup, \Rightarrow, \neg}, \ AX, (MP) \dfrac{A, (A \Rightarrow B)}{B})
\]
that is complete under classical classical semantics.

Definition 1 Let \( X \subseteq F \) be any subset of the set \( F \) of formulas of the language \( \mathcal{L}_{\cap, \cup, \Rightarrow, \neg} \) of \( S \).
We define a set \( Cn(X) \) of all consequences of the set \( X \) as follows
\[
Cn(X) = \{ A \in F : X \vdash_S A \},
\]
i.e. \( Cn(X) \) is the set of all formulas that can be proved in \( S \) from the set \( (AX \cup X) \). The following theorem holds for \( S \).

Part 1 (5pts)
(i) Prove that for any subsets \( X, Y \) of the set \( F \) of formulas the following monotonicity property holds.
If \( X \subseteq Y \), then \( Cn(X) \subseteq Cn(Y) \)

(ii) Prove that for any \( X \subseteq F \), the set \( T \) of all propositional classical tautologies is a subset of \( Cn(X) \), i.e.
\[
T \subseteq Cn(X).
\]
Part two (15pts) Prove that for any $A, B \in F$, $X \subseteq F$,

$$(A \cap B) \in Cn(X) \iff A \in Cn(X) \text{ and } B \in Cn(X)$$

**Hint:** Use the Monotonicity and Completeness of $S$ i.e. the fact that any tautology you might need is provable in $S$. 
Part Three: (15pts) Prove that for any $A, B \in F$,

$$Cn\{\{A, B\}\} = Cn\{\{A \cap B\}\}$$

**Hint:** Use Deduction Theorem and Completeness of $S$. 

7
QUESTION 5 (20pts) Given a tautology $A$, and the set $V_A$ of all truth assignment restricted to $A$, the Proof 1 of the Completeness Theorem for the system $S$ defines a method of efficiently combining $v \in V_A$ to construct a proof of $A$ in $S$.

Let consider the following tautology $A = A(a, b)$

$$A = ((a \Rightarrow b) \Rightarrow (\neg b \Rightarrow \neg a))$$

Write down all steps of the construction of the proof of $A$ as described in the Proof 1 with justification why they are correct.