CSE541 Midterm 1 Spring 2011
100 points

NAME **ID:**

QUESTION 1 (10pts)

1. Use Dedekind theorem to prove that the set R of real numbers is infinite.

2. Find a function f that is $1-1$ and maps R ONTO $R - \{1, 8, 10\}$.

QUESTION 2 (20pts)

Here are some definitions; some of them are known to you and put as a reminder.

Definition 1 By a **m-valued semantics** S_m for a propositional language $\mathcal{L} = \mathcal{L}_{\{\neg,\cap,\cup,\Rightarrow\}}$ we understand any definition of of connectives $\neg, \cap, \cup, \Rightarrow$ as operations on a set $V_m = \{v_1, v_2, ...v_m\}$ of logical values.
We assume that $v_1 \leq v_2 \leq ... \leq v_m$, i.e. V_m is totally ordered by a certain relation $\leq$ with v_1, v_m being smallest and greatest elements, respectively. We denote $v_1 = F$, $v_m = T$ and call them (total) False and Truth, respectively.

Definition 2 Let VAR be a set of propositional variables of $\mathcal{L}$ and let S_m be any m-valued semantics for $\mathcal{L}$. A truth assignment $v : VAR \longrightarrow V_m$ is called **a** S_m **model** for a formula A of $\mathcal{L}$ iff $v(A) = T$ and logical value $v(A)$ is evaluated accordingly to the semantics S_m. We denote is symbolically as

$$v \models_{S_m} A.$$

Any v such that v **is not a** S_m **model** for a formula A is called **a counter-model** for A.

Definition 3 A formula A of $\mathcal{L}$ is called **a** S_m **tautology** iff $v \models_{S_m} A$, for all v. We denote it by $\models_{S_m} A$, and $\models A$ for **classical semantics tautologies**.

Definition 4 A proof system S is **complete** with respect to a semantics S_m iff for any formula A,the following holds:

A is provable in S iff A is S_m tautology.

Q2 Part one (15pts)

Let S_3 be a 3-valued semantics for $\mathcal{L}_{\{\neg,\cap,\cup,\Rightarrow\}}$ defined as follows.
$V_3 = \{F, U, T\}$, for $F \leq U \leq T$ and

$\cup$	F	U	T
F	F	U	T
U	U	U	U
T	T	U	T

$\neg$	F	U	T
	T	F	U

$a \cup b = min\{a, b\}$, $\quad a \Rightarrow b = \neg a \cup b$, for any $a, b \in V_3$.

Consider the following classical tautologies:

$$A_1 = (A \cup \neg A), \qquad A_2 = (A \Rightarrow (B \Rightarrow A)).$$

(a) Find S_3 counter-models for A_1, A_2, if exist. Use shorthand notation.

(b) Define a 2-valued semantics S_2 for $\mathcal{L}$, such that **none of** A_1, A_2 is a S_2 tautology. Verify your results. Use shorthand notation.

(c) Define a 3-valued semantics C_3 for $\mathcal{L}$, such that both A_1, and A_2 are a C_3 tautologies. Verify your results. Use shorthand notation.

Q2 Part Two (5pts)

Let $S = (\mathcal{L}, \mathbf{A1}, \mathbf{A2}, \mathbf{A3}, MP)$ be a proof system with axioms:

A1 $(A \Rightarrow (B \Rightarrow A))$,

A2 $((A \Rightarrow (B \Rightarrow C)) \Rightarrow ((A \Rightarrow B) \Rightarrow (A \Rightarrow C)))$,

A3 $((\neg B \Rightarrow \neg A) \Rightarrow ((\neg B \Rightarrow A) \Rightarrow B))$,

The system S is complete with respect to classical semantics.

Verify whether S is complete with respect to 3-valued semantics S_3, as defined at the beginning of this question.

QUESTION 3 (15pts)

Let S be from QUESTION 2, PART 2.

The following Lemma holds in the system S.

LEMMA For any $A, B, C \in \mathcal{F}$,

(a) $(A \Rightarrow B), (B \Rightarrow C) \vdash_H (A \Rightarrow C)$,

(b) $(A \Rightarrow (B \Rightarrow C)) \vdash_H (B \Rightarrow (A \Rightarrow C))$.

Complete the proof sequence (in S)

$$B_1, ..., B_9$$

by providing comments how each step of the proof was obtained.

$B_1 = (A \Rightarrow B)$

$B_2 = (\neg\neg A \Rightarrow A)$

Already PROVED

$B_3 = (\neg\neg A \Rightarrow B)$

$B_4 = (B \Rightarrow \neg\neg B)$

Already PROVED

$B_5 = (\neg\neg A \Rightarrow \neg\neg B)$

$B_6 = ((\neg\neg A \Rightarrow \neg\neg B) \Rightarrow (\neg B \Rightarrow \neg A))$

Already PROVED

$B_7 \quad (\neg B \Rightarrow \neg A)$

$B_8 \quad (A \Rightarrow B) \vdash (\neg B \Rightarrow \neg A)$

$B_9 = ((A \Rightarrow B) \Rightarrow (\neg B \Rightarrow \neg A))$

QUESTION 4 (35pts)

Consider any proof system S,

$$S = (\mathcal{L}_{\{\cap, \cup, \Rightarrow, \neg\}},\ AX,\ (MP)\frac{A, (A \Rightarrow B)}{B})$$

that is **complete** under classical classical semantics.

Definition 1 Let $X \subseteq F$ be any subset of the set F of formulas of the language $\mathcal{L}_{\{\cap, \cup, \Rightarrow, \neg\}}$ of S.

We define a set $Cn(X)$ of all **consequences** of the set X as follows

$$Cn(X) = \{A \in F :\ X \vdash_S A\},$$

i.e. $Cn(X)$ is the set of all formulas that can be proved in S from the set $(AX \cup X)$. The following theorem holds for S.

Part 1 (5pts)

(i) Prove that for any subsets X, Y of the set F of formulas the following **monotonicity property** holds.

If $X \subseteq Y$, then $Cn(X) \subseteq Cn(Y)$

(ii) Prove that for any $X \subseteq F$, the set $\mathbf{T}$ of all propositional classical tautologies is a subset of $Cn(X)$, i.e.

$$\mathbf{T} \subseteq Cn(X).$$

Part two (15pts) Prove that for any $A, B \in F$, $X \subseteq F$,

$$(A \cap B) \in Cn(X) \;\; iff \;\; A \in Cn(X) \; and \; B \in Cn(X)$$

Hint: Use the Monotonicity and Completeness of S i.e. the fact that any tautology you might need is provable in S.

Part Three: (15pts) Prove that for any $A, B \in F$,

$$Cn(\{A, B\}) = Cn(\{(A \cap B)\})$$

Hint: Use Deduction Theorem and Completeness of S.

QUESTION 5 (20pts) Given a tautology A, and the set V_A of all truth assignment restricted to A, the Proof 1 of the Completeness Theorem for the system S defines a method of efficiently combining $v \in V_A$ to construct a proof of A in S.

Let consider the following tautology $A = A(a, b)$

$$A = ((a \Rightarrow b) \Rightarrow (\neg b \Rightarrow \neg a))$$

Write down all steps of the construction of the proof of A as described in the Proof 1 with justification why they are correct.