GENERAL INFORMATION
The webpage contains
All Book Chapters,
All lectures slides
Homework and Sample Problems with solutions item
Sample quizzes and Tests
Course Text Book

AN INTRODUCTION TO CLASSICAL and NON-CLASSICAL LOGICS
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Full Book Text and Lecture Slides are in Downloads on the course web page

Additional Book:
Chapter 1
INTRODUCTION

PART 1: Mathematical Paradoxes
PART 2: Logics for Computer Science
Chapter 1
PART ONE: Mathematical Paradoxes

Early Intuitive Approach:
Until recently, till the end of the 19th century, mathematical theories used to be built in the intuitive, or axiomatic way. Historical development of mathematics has shown that it is not sufficient to base theories only on an intuitive understanding of their notions.
Example

Consider the following.

By a set, we mean intuitively, any collection of objects. For example, the set of all even integers or the set of all students in a class.

The objects that make up a set are called its members (elements).

Sets may themselves be members of sets for example, the set of all sets of integers has sets as its members.
Example

Sets may themselves be members of sets for example, the set of all sets of integers has sets as its members.
Most sets are not members of themselves; the set of all students, for example, is not a member of itself, because the set of all students is not a student.
However, there may be sets that do belong to themselves - for example, the set of all sets.
Russell Paradox, 1902

Russell Paradox
Consider the set $A$ of all those sets $X$ such that $X$ is not a member of $X$.

Clearly, $A$ is a member of $A$ if and only if $A$ is not a member of $A$.

So, if $A$ is a member of $A$, the $A$ is also not a member of $A$; and if $A$ is not a member of $A$, then $A$ is a member of $A$.

In any case, $A$ is a member of $A$ and $A$ is not a member of $A$.

CONTRADICTION!
Russell Paradox Solution

Russel proposed his Theory of Types as a solution to the Paradox.

The idea is that every object must have a definite non-negative integer as its type assign to it.

An expression $x$ is a member of the set $y$ is meaningful if and only if the type of $y$ is one greater than the type of $x$. 
Russell Paradox Solution

Russell’s theory of types guarantees that it is meaningless to say that a set belongs to itself. Hence Russell’s solution is: The set $A$ as stated in the Russell paradox does not exist. The Type Theory was extensively developed by Whitehead and Russell in years 1910 - 1913. It is successful, but difficult in practice and has certain other drawbacks as well.
LOGICAL ANTINOMIES

Paradoxes concerning the notion of a set are called Logical Paradoxes, or Logical Antinomies.

A Modern development of Axiomatic Set Theory as one of the most important fields of Modern Mathematics or more specifically Mathematical Logic, or Foundations of Mathematics resulted from the search for solutions to various Logical Paradoxes.

Zermello developed in 1908 a first paradoxes free axiomatic set theory.
LOGICAL PARADOXES

Two of the most known Paradoxes, other then Russell’s Paradox are Cantor and Burali-Forti antinomies. They were stated at the end of 19th century. Cantor Paradox involves the theory of cardinal numbers. Burali-Forti Paradox is the analogue to Cantor’s but in the theory of ordinal numbers.
Cardinality of Sets

We say that sets $X$ and $Y$ have the same cardinality, $\text{card}X = \text{card}Y$ or that they are equinumerous if and only if there is one-to-one correspondence that maps $X$ and $Y$

$\text{card}X \leq \text{card}Y$ means that $X$ is equinumerous with a subset of $Y$

$\text{card}X < \text{card}Y$ means that $\text{card}X \leq \text{card}Y$ and $\text{card}X \neq \text{card}Y$
Cantor and Schröder-Berstein Theorems

Cantor Theorem
For any set $X$, 
$\text{card}X < \text{card}\mathcal{P}(X)$

Schröder-Berstein Theorem
For any sets $X$ and $Y$, 
If $\text{card}X \leq \text{card}Y$ and $\text{card}Y \leq \text{card}X$, then $\text{card}X = \text{card}Y$.

Ordinal numbers are the numbers assigned to sets in a similar way as cardinal numbers but they deal with ordered sets.
Cantor Paradox, 1899

Let $C$ be the universal set - that is, the set of all sets. Now, $\mathcal{P}(C)$ is a subset of $C$, so it follows easily that

$$\text{card}\mathcal{P}(C) \leq \text{card}C.$$ 

On the other hand, by Cantor Theorem,

$$\text{card}C < \text{card}\mathcal{P}(C) \leq \text{card}\mathcal{P}(C)$$

so also

$$\text{card}C \leq \text{card}\mathcal{P}(C).$$

From Schröder- Berstein theorem we have that $\text{card}\mathcal{P}(C) = \text{card}C$, what contradicts Cantor Theorem. Solution: Universal set does not exist.
Burali-Forti Paradox, 1897

Given any ordinal number, there is a still larger ordinal number. But the ordinal number determined by the set of all ordinal numbers is the largest ordinal number.

Solution: the set of all ordinal numbers do not exist.
Semantic Paradoxes

Another solution to Semantic Paradoxes:
Reject the assumption that for every property $P(x)$, there exists a corresponding set of all objects $x$ that satisfy $P(x)$.
Russell’s Paradox then simply proves that there is no set $A$ for $P(x)$: of all sets that do not belong to themselves.
Cantor Paradox shows that there is no set $A$ for $P(x)$: there is universal set
Burali-Forti Paradox shows that there is no set $A$ for $P(x)$: there is a set that contains all ordinal numbers.
Intuitionism

A more radical interpretation of the paradoxes has been advocated by Brower and his intuitionist school. Intuitionists refuse to accept the universality of certain basic logical laws, such as the law of excluded middle: A or not A. For intuitionists the excluded middle law is true for finite sets, but it is invalid to extend it to all sets.

The intuitionists’ concept of infinite set differs from that of classical mathematicians.
Intuitionism

Infinite set for the intuitionists is something which is constantly in a state of formation.

Example: set of positive integers is infinite because to any given finite set of positive integers it is always possible to add one more positive integer.

Also, for intuitionists the notion of the set of all subsets of the set of all positive integers is not regarded meaningful.
Intuitionists’ Mathematics

The basic difference between classical and intuitionists’ mathematics lies in the interpretation of the word exists.

Example: let $P(n)$ be a statement in the arithmetic of positive integers.

For the (classical) mathematician the sentence there exists $n$, such that $P(n)$ is true if it can be deduced (proved) from the axioms of arithmetic by means of classical logic.

In classical mathematics proving existence of $x$ does not mean that one is able to indicate a method of construction of a positive integer $n$ such that $P(n)$ holds.
[In general in the intuitionists’ universe we are justified in asserting the existence of an object having a certain property only if we know an effective method for constructing, or finding such an object.

In intuitionist’ mathematics the paradoxes are not derivable, or even meaningful.

The Intuitionism, because of its constructive flavor, has found a lot of applications in computer science.

For example in the theory of programs correctness.

Intuitionistic Logic (to be studied in this course) reflects intuitionists ideas in a form a formalized deductive system.
The development of axiomatic theories solved some but not all problems brought up by the Logical Paradoxes. Even the consistent sets of axioms, as the following examples show, do not prevent the occurrence of another kind of paradoxes, called Semantic Paradoxes.
Berry Paradox, 1906: Let $A$ denote the set of all positive integers which can be defined in the English language by means of a sentence containing at most 1000 letters. The set $A$ is finite since the set of all sentences containing at most 1000 letters is finite. Hence, there exist positive integer which do not belong to $A$.

Consider a sentence: $n$ is the least positive integer which cannot be defined by means of a sentence of the English language containing at most 1000 letters.

This sentence contains less than 1000 letters and defines a positive integer $n$.

Therefore $n \in A$. But $n \notin A$ by the definition of $n$. CONTRADICTION!
Berry Paradox Analysis

The paradox resulted entirely from the fact that we did not say precisely what notions and sentences belong to the arithmetic and what notions and sentences concern the arithmetic. Of corse we talk about and examine arithmetic as a fix and closed deductive system. And on the top of it we also incorrectly mixed the natural language with mathematical language.
Berry Paradox Solution

We have to distinguish always the language of the theory (arithmetic) and the language which talks about the theory, called a metalanguage.

In general we must distinguish a theory from the meta-theory. In well and correctly defined theory the such paradoxes can not appear.
The Liar Paradox

A man says: I am lying.
If he is lying, then what he says is true, and so he is not lying.
If he is not lying, then what he says is not true, and so he is lying.
CONTRADICTION!
Löb Paradox, 1955

Let $A$ be any sentence.
Let $B$ be a sentence: If this sentence is true, then $A$.
So, $B$ asserts: If $B$ is true then $A$.
Now consider the following argument:
Assume $B$ is true.
Then, by $B$, since $B$ is true, $A$ is true.
This argument shows that, if $B$ is true, then $A$.
But this is exactly what $B$ asserts. Hence, $B$ is true.
Therefore, by $B$, since $B$ is true, $A$ is true.
Thus every sentence is true - CONTRADICTION!
Liar Paradoxes

These paradoxes arise because the concepts of ”I am true”, ”this sentence is true”, ”I am lying” should not occur in the language of the theory.

They belong to a metalanguage of the theory, it means a language that talks about the theory.
The Liar Paradox is a corrected version of a following paradox stated in antiquity by a Cretan philosopher Epimenides.

The Cretan philosopher Epimenides said: All Cretans are liars.

If what he said is true, then, since Epimenides is a Cretan, it must be false.

Hence, what he said is false. Thus, there is a Cretan who is not a liar.

CONTRADICTION with what he said: ”All Cretans are liars”.
GENERAL REMARKS;
The Goal of the Course

FIRST TASK when one builds mathematical logic foundations of mathematics or of computer science is to define formally and proper symbolic language. This is called building a proper syntax. SECOND TASK is to extend the syntax to include a notion of a proof. It allows us to find out what can and cannot be proved if certain axioms and rules of inference are assumed. This part of syntax is called PROOF THEORY.
GENERAL REMARKS;
The Goal of the Course

THIRD TASK is to define formally what does it mean that formulas of our formal language defined in the TASK ONE are true, i.e. to define what we formally call a semantics for our language.

For example the notion of truth i.e. the semantics for classical and intuitionistic approaches are different.

FOUTH TASK is to investigate the relationship between proof theory (part of the syntax) and semantics.

The GOAL of this course is to follow the above Four Tasks in case of the Classical Logic (and hence classical mathematics) and in case of some non-classical logics.