CSE541 EXERCISE 9(1)

QUESTION 1

$H$ is the following proof system:

$H = (\mathcal{L}_{\Rightarrow, \neg}, \mathcal{F}, AX = \{A1, A2, A3\}, MP)$

A1  $(B \Rightarrow (A \Rightarrow B)),$
A2  $(B \Rightarrow \neg \neg B),$
A3  $(A \Rightarrow (\neg B \Rightarrow \neg (A \Rightarrow B))),$
A4  $(\neg A \Rightarrow (A \Rightarrow B)),$
MP (Rule of inference)

\[
(\text{MP}) \quad \frac{A \ (A \Rightarrow B)}{B}
\]

(1) Prove that $H$ is SOUND under classical semantics.

(2) Prove that The Main Lemma of Chapter 9 HOLDS for $H$.

The Main Lemma states:

For any formula $A$ and a variable assignment $v$, if $A', B_1, B_2, ..., B_n$ are corresponding formulas defined by 0.1, then

$B_1, B_2, ..., B_n \vdash A'$.

Definition 0.1 Let $A$ be a formula and $b_1, b_2, ..., b_n$ be all propositional variables that occur in $A$. Let $v$ be variable assignment $v : \text{VAR} \rightarrow \{T, F\}$. We define, for $A, b_1, b_2, ..., b_n$ and $v$ a corresponding formulas $A', B_1, B_2, ..., B_n$ as follows:

\[
A' = \begin{cases} 
A & \text{if } v^*(A) = T \\
\neg A & \text{if } v^*(A) = F
\end{cases}
\]

\[
B_i = \begin{cases} 
b_i & \text{if } v(b_i) = T \\
\neg b_i & \text{if } v(b_i) = F
\end{cases}
\]

for $i = 1, 2, ..., n$.

QUESTION 2 Here is a fragment of the proof of the Main Lemma (chapter 9).
Case: A is \( (A_1 \Rightarrow A_2) \)

If A is of the form \( (A_1 \Rightarrow A_2) \) then \( A_1 \) and \( A_2 \) have less than \( n \) connectives and so by the inductive assumption we have \( B_1, B_2, ..., B_n \vdash A_1' \) and \( B_1, B_2, ..., B_n \vdash A_2' \), where \( B_1, B_2, ..., B_n \) are formulas corresponding to the propositional variables in A.

Write down all steps that justify correctness of the above statement.

**QUESTION 3** Give examples of 3 proof systems \( S_1, S_2, S_3 \) with some sets of axioms that make them complete with respect to classical semantics. Two of them must have languages containing more connectives then \( \neg, \Rightarrow \).

Justify why and how you can carry the proof of completeness theorem for each of them.

**QUESTION 4** Here is a more detailed version of our Proof 1 of the Completeness Theorem (as in new Chapter 9)

Assume that \( \models A \). Let \( b_1, b_2, ..., b_n \) be all propositional variables that occur in A, i.e. \( A = A(b_1, b_2, ..., b_n) \).

By the Main Lemma we know that, for any variable assignment \( v \), the corresponding formulas \( A', B_1, B_2, ..., B_n \) can be found such that \( B_1, B_2, ..., B_n \vdash A' \).

Note here that \( A' \) of the definition is \( A \), since since \( \models A \).

Let \( v : VAR \rightarrow \{T, F\} \) be any variable assignment, and

\[ v_A : \{b_1, b_2, ..., b_n\} \rightarrow \{T, F\} \quad (1) \]

its restriction to the formula A. We get by the Main Lemma that the \( v_A \) (1) and the formula A define the appropriate formulas \( B_1, B_2, ..., B_n \) such that

\[ B_1, B_2, ..., B_n \vdash A. \quad (2) \]

The proof is based on a method of constructive elimination of all hypothesis \( B_1, B_2, ..., B_n \) in 2) to finally construct the proof of \( A \) in \( S \) i.e. we prove that \( \vdash A \).

**Step 1: elimination** of \( B_n \). The form of \( B_n \) depends of the logical values assigned to \( b_n \) by the \( v_A \) (1), so we have to consider 2 cases: \( v_A(b_n) = T \) or \( v_A(b_n) = F \).

**Case 1:** \( v_A(b_n) = T \) and then by definition \( B_n = b_n \) and by the Main Lemma

\[ B_1, B_2, ..., b_n \vdash A. \]

By Deduction Theorem we have that

\[ B_1, B_2, ..., B_{n-1} \vdash (b_n \Rightarrow A). \quad (3) \]

**Case 2:** \( v_A(b_n) = F \) and hence \( B_n = \neg b_n \). By the Main Lemma

\[ B_1, B_2, ..., B_{n-1}, \neg b_n \vdash A. \]
By the Deduction Theorem we have that
\[ B_1, B_2, \ldots, B_{n-1} \vdash (\neg b_n \Rightarrow A). \] (4)

By the assumed provability of the formula 9 for \( A = b_n, B = A \) we have that
\[ \vdash ((b_n \Rightarrow A) \Rightarrow ((\neg b_n \Rightarrow A) \Rightarrow A)). \]

By monotonicity we have that
\[ B_1, B_2, \ldots, B_{n-1} \vdash ((b_n \Rightarrow A) \Rightarrow ((\neg b_n \Rightarrow A) \Rightarrow A)). \] (5)

Applying Modus Ponens twice to the above property 5 and properties (3), (4) we get that
\[ B_1, B_2, \ldots, B_{n-1} \vdash A. \] (6)

and hence we have eliminated \( B_n \).

**Step 2: elimination of \( B_{n-1} \).** We repeat the Step 1.

As before we have 2 cases to consider: \( v_A(b_{n-1}) \) may be T or F. In both cases we apply Main Lemma, Deduction Theorem, monotonicity, proper substitutions of assumed provability of the formula 9 i.e the fact that \( \vdash ((A \Rightarrow B) \Rightarrow ((\neg A \Rightarrow B) \Rightarrow B)) \), and Modus Ponens twice and eliminate \( B_{n-1} \) just as we eliminated \( B_n \).

**After n steps,** we finally obtain that
\[ \vdash A. \]

**Write down** following the STEP 1, all detailed steps of the Step 2: elimination of \( B_{n-1} \).

**QUESTION 5** Apply the above proof step by step to construct proof in \( S \) of the following de Morgan Law.
\[ (\neg(a \cup b) \Rightarrow (\neg a \cap \neg b)). \]

**Hint** the proof 1 works only for the language \( L_{(\Rightarrow, \neg)}. \)