Problem 1

Given a proof system:

\[ S = (\mathcal{L}_{\neg, \rightarrow}, \ \mathcal{E} = \mathcal{F}, AX = \{(A \rightarrow A), (A \rightarrow (\neg A \rightarrow B))\}, \ (r) \ \frac{(A \rightarrow B)}{(B \rightarrow (A \rightarrow B))}. \]

1. Prove that \( S \) is sound under classical semantics.

2. Prove that \( S \) is not sound under \( \mathbf{K} \) semantics.

3. Write a formal proof in \( S \) with 2 applications of the rule \( (r) \).

Solution of 1.

**Definition:** System \( S \) is sound if and only if

(i) Axioms are tautologies and

(ii) rules of inference are sound, i.e lead from all true premisses to a true conclusion.

We verify the conditions (i), (ii) of the definition as follows.

(i) Both axioms of \( S \) are basic classical tautologies.

(ii) Consider the rule of inference of \( S \).

\( (r) \ \frac{(A \rightarrow B)}{(B \rightarrow (A \rightarrow B))}. \)

Assume that its premise (the only premise) is True, i.e. let \( v \) be any truth assignment, such that \( v^*(A \rightarrow B) = T \). We evaluate logical value of the conclusion under the truth assignment \( v \) as follows.

\[ v^*(B \rightarrow (A \rightarrow B)) = v^*(B) \rightarrow v^*(A \rightarrow B) = T \]

for any \( B \) and any value of \( v^*(B) \).

**Solution of 2.** System \( S \) is not sound under \( \mathbf{K} \) semantics because axiom \( (A \rightarrow A) \) is not a \( \mathbf{K} \) semantics tautology.

**Solution of 3.** There are many solutions. Here is one of them.

Required formal proof is a sequence \( A_1, A_2, A_3 \), where

\( A_1 = (A \rightarrow A) \)  
(Axiom)

\( A_2 = (A \rightarrow (A \rightarrow A)) \)

Rule \( (r) \) application 1 for \( A = A, B = A \).

\( A_3 = ((A \rightarrow A) \rightarrow (A \rightarrow (A \rightarrow A))) \)

Rule \( (r) \) application 2 for \( A = A, B = (A \rightarrow A) \).
Problem 2

Prove, by constructing a formal proof that
\[ \vdash_S ((\neg A \Rightarrow B) \Rightarrow (A \Rightarrow (\neg A \Rightarrow B))), \]
where \( S \) is the proof system from Problem 1.

Solution: Required formal proof is a sequence \( A_1, A_2 \), where
\[ A_1 = (A \Rightarrow (\neg A \Rightarrow B)) \]
Axiom
\[ A_2 = ((\neg A \Rightarrow B) \Rightarrow (A \Rightarrow (\neg A \Rightarrow B))) \]
Rule (r) application for \( A = A, B = (\neg A \Rightarrow B) \).

Observe that we needed only one application of the rule (r). One more application of the rule (r) to \( A_2 \) gives another solution to Problem 1, namely a proof \( A_1, A_2, A_3 \) for \( A_1, A_2 \) defined above and
\[ A_3 = ((A \Rightarrow (\neg A \Rightarrow B)) \Rightarrow (A \Rightarrow (\neg A \Rightarrow B))) \]
Rule (r) application for \( A = (\neg A \Rightarrow B) \) and \( B = (A \Rightarrow (\neg A \Rightarrow B)) \).

Problem 3

Given a proof system:
\[ S = (\mathcal{L}_{\{\cup, \Rightarrow\}}, \mathcal{E} = \mathcal{F}, AX = \{A_1, A_2\}, \mathcal{R} = \{(r)\}), \]
where
\[ A_1 = (A \Rightarrow (A \cup B)), \quad A_2 = (A \Rightarrow (B \Rightarrow A)) \]
and
\[ (r) \begin{align*}
(A \Rightarrow B) & \quad \frac{(A \Rightarrow (B \Rightarrow A))}{(A \Rightarrow B)}
\end{align*} \]

Prove that \( S \) is sound under classical semantics.

Solution: Axioms of \( S \) are basic classical tautologies. The proof of soundness of the rule of inference is the following.
Assume \( (A \Rightarrow B) = T \). Hence the logical value of conclusion is \( (A \Rightarrow (A \Rightarrow B)) = (A \Rightarrow T) = T \) for all \( A \).

Problem 4

Determine whether \( S \) from the Problem 3 is sound or not sound under \( K \) semantics.

Solution 1: \( S \) is not sound under \( K \) semantics. Let’s take truth assignment such that \( A = \bot, B = \bot \). The logical value of axiom \( A_1 \) is as follows.
\[ (A \Rightarrow (A \cup B)) = (\bot \Rightarrow (\bot \cup \bot)) = \bot \quad \text{and} \quad \not\models_K (A \Rightarrow (A \cup B)). \]

Observe that the \( v \) such that \( A = \bot, B = \bot \) is not the only \( v \) that makes \( A_1 \neq T \), i.e. proves that \( \not\models_K A_1 \).
\( (A \Rightarrow (A \cup B)) \neq T \) if and only if \( (A \Rightarrow (A \cup B)) = F \) or \( (A \Rightarrow (A \cup B)) = \bot \). The first case is impossible because \( A_1 \) is a classical tautology.
Consider the second case. \( (A \Rightarrow (A \cup B)) = \bot \) in two cases.
c1 \( A = \bot \) and \((A \cup B) = F\), i.e. \((\bot \cup B) = F\), what is impossible.

\( c2 \) \( A = T \) and \((A \cup B) = \bot\), i.e. \((T \cup B) = \bot\), what is impossible.

\( c3 \) \( A = \bot \) and \((A \cup B) = \bot\), i.e. \((\bot \cup B) = \bot\). This is possible for \( B = \bot \) or \( B = F \), i.e when \( A = \bot, B = \bot \) or \( A = \bot, B = F \).

From the above Observation we get second solution.

Solution 2: \( S \) is not sound under \( K \) semantics. Axiom \( A1 \) is not \( K \) semantics tautology. There are exactly two truth assignments \( v \), such that \( v \not\models A1 \). One is, as defined in Solution 1: \( A = \bot, B = \bot \). The second is \( A = \bot, B = F \).

**Problem 5**

Write a formal proof \( A_1, A_2, A_3 \) in \( S \) from the Problem 3 with 2 applications of the rule \((r)\) that starts with axiom \( A1 \), i.e such that \( A1 = A1 \).

Solution: The formal proof \( A_1, A_2, A_3 \) is as follows.

\[ A_1 = (A \Rightarrow (A \cup B)) \]

Axiom

\[ A_2 = (A \Rightarrow (A \Rightarrow (A \cup B))) \]

Rule \((r)\) application for \( A = A \) and \( B = (A \cup B) \)

\[ A_3 = (A \Rightarrow (A \Rightarrow (A \Rightarrow (A \cup B)))) \]

Rule \((r)\) application for \( A = A \) and \( B = (A \Rightarrow (A \cup B)) \).

**Problem 6**

Use results from Problem 4 to determine whether \( \models_K A_3 \).

**Solution 1:** We use the two \( v \) from QUESTION 3 to evaluate the logical value of \( A_3 \). Namely we evaluate:
\[ v^*(A \Rightarrow (A \Rightarrow (A \Rightarrow (A \Rightarrow (A \Rightarrow F)))) = \bot \text{, or } v^*(A \Rightarrow (A \Rightarrow (A \Rightarrow (A \Rightarrow (A \Rightarrow F)))) = \bot \text{. Both cases prove that } \not\models_K A_3 \text{.} \]

**Solution 2:** We know that \( S \) is not sound, because there is \( v \) for which \( A1 = A_1 = \bot \), as evaluated in Problem 4. We prove that the rule \((r)\) preserves the logical value \( \bot \) under any \( v \) such that \( A1 = \bot \), as follows.

Let the premiss \((A \Rightarrow B) = \bot\), the logical value of the conclusion is hence \( (A \Rightarrow \bot) = \bot \) for \( A = \bot, T \) and \( (A \Rightarrow \bot) = T \) for \( A = F \).

The case \( A = F \) evaluates the premiss \((A \Rightarrow B) = (F \Rightarrow B) = T \) for all \( B \), what contradicts the assumption that \((A \Rightarrow B) = \bot\). We must hence have \( A = \bot \). But all possible \( v \) for which \( A1 = \bot \) are such that \( A = \bot \), what end the proof.

It means that any \( A \) such that \( A \) has proof that starts with axiom \( A1 \) and then multiple applications of the rule \((r)\) is evaluated to \( \bot \) under all \( v \), such that \( v^*(A1) = \bot \). Hence, in particular, \( \not\models_K A_3 \).

**Problem 7**

Write a formal proof \( A_1, A_2 \) in \( S \) from the Problem 3 with 1 application of the rule \((r)\) that starts with axiom \( A2 \), i.e such that \( A1 = A2 \).
Solution: The formal proof $A_1, A_2$ is as follows.

$A_2 = (A \Rightarrow (B \Rightarrow A))$

Axiom

$A_2 = (A \Rightarrow (A \Rightarrow (B \Rightarrow A)))$

Rule ($r$) application for $A = A$ and $B = (B \Rightarrow A)$.

Problem 8

Use results from Problem 3 to determine whether $\models A_2$.

Solution: System $S$ is sound under classical semantics, hence by the Soundness Theorem we get that $\models (A \Rightarrow (A \Rightarrow (B \Rightarrow A)))$, as it has a proof in $S$.

Problem 9

Prove, by constructing a formal proof in $S$ from the Problem 3 that

$\vdash_S (A \Rightarrow (A \Rightarrow (A \Rightarrow (A \Rightarrow A))))$.

Solution: $A_2 = (A \Rightarrow (A \Rightarrow A))$

Axiom for $B = A$

$A_2 = (A \Rightarrow (A \Rightarrow (A \Rightarrow A)))$

Rule ($r$) application for $A = A$ and $B = (A \Rightarrow A)$.

$(A \Rightarrow (A \Rightarrow (A \Rightarrow (A \Rightarrow A))))$

Rule ($r$) application for $A = A$ and $B = (A \Rightarrow (A \Rightarrow A))$.  

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