SOLVE ALL PROBLEMS as PRACTICE

FINITE and INFINITE SETS

Definition 1 A set \( A \) is FINITE iff there is a natural number \( n \in N \) and there is a 1−1 function \( f \) that maps the set \( \{1, 2, \ldots, n\} \) onto \( A \).

Definition 2 A set \( A \) is INFINITE iff it is NOT FINITE.

QUESTION 1 Use the above definition to prove the following

FACT 1 A set \( A \) is INFINITE iff it contains a countably infinite subset, i.e. one can define a 1−1 sequence \( \{a_n\}_{n \in N} \) of some elements of \( A \).

Definition 3 Two sets \( A, B \) have the same CARDINALITY iff there is a function \( f \) that maps \( A \) one-to-one onto the set \( B \). We denote it \( |A| = |B| = M \) and \( M \) is called a cardinal number of sets \( A \) and \( B \).

QUESTION 2 Use the above definition and FACT 1 from Question 1 to prove the following characterization of infinite sets.

Dedekind Theorem A set \( A \) is INFINITE iff there is a set proper subset \( B \) of the set \( A \) such that \( |A| = |B| \).

QUESTION 3 Use technique from DEDEKIND THEOREM to prove the following

Theorem For any infinite set \( A \) and its finite subset \( B \), \( |A| = |A - B| \).

QUESTION 4 Use DEDEKIND THEOREM to prove that the set \( N \) of natural numbers is infinite.

QUESTION 5 Use DEDEKIND THEOREM to prove that the set \( R \) of real numbers is infinite.

QUESTION 6 Use technique from DEDEKIND THEOREM to prove that the interval \( [a, b], a < b \) of real numbers is infinite and that \( |[a, b]| = |(a, b)| \).

CARDINALITIES OF SETS

Definition 4 For any sets \( A, B \), let \( |A| = \mathcal{N} \) and \( |B| = \mathcal{M} \). We say \( \mathcal{N} \leq \mathcal{M} \) iff \( |A| = |C| \) for some \( C \subseteq B \). We say \( \mathcal{N} < \mathcal{M} \) iff \( \mathcal{N} \leq \mathcal{M} \) and \( \mathcal{N} \neq \mathcal{M} \).

QUESTION 7 Prove, using the above definitions 3 and 4 that for any cardinal numbers \( \mathcal{M}, \mathcal{N}, \mathcal{K} \) the following formulas hold:

- \( \mathcal{N} \leq \mathcal{N} \)
- \( 2. \text{If } \mathcal{N} \leq \mathcal{M} \text{ and } \mathcal{M} \leq \mathcal{K}, \text{ then } \mathcal{N} \leq \mathcal{K} \).

QUESTION 8 Prove, for any sets \( A, B, C \) the following holds.
Fact 2

If $A \subseteq B \subseteq C$ and $|A| = |C|$, then $|A| = |B|$.

To prove $|A| = |B|$ you must use definition 3, i.e to construct a proper function. Use the construction from proofs of Fact 1 and Question 3.

QUESTION 9 Prove the following

Berstein Theorem (1898) For any cardinal numbers $\mathcal{M}, \mathcal{N}$

$\mathcal{N} \leq \mathcal{M}$ and $\mathcal{M} \leq \mathcal{N}$ then $\mathcal{N} = \mathcal{M}$.

1. Prove first the case when the sets $A, B$ are disjoint.

2. Generalize the construction for 1. to the not-disjoint case.

REMINDER

Definition 5 A set $A$ is INFINITELY COUNTABLE iff $A$ has the same cardinality as Natural numbers $\mathbb{N}$, i.e. $|A| = |\mathbb{N}| = \aleph_0$

Definition 6 A set $A$ is COUNTABLE iff $A$ is finite or infinitely countable.

Definition 7 A set $A$ is UNCOUNTABLE iff $A$ is NOT countable.