CHAPTER 9

Two Proofs of Completeness Theorem

There are many proof systems that describe classical propositional logic, i.e. that are complete proof systems with the respect to the classical semantics.

We present here a Hilbert proof system for the classical propositional logic and discuss two ways of proving the Completeness Theorem for it.

Any proof of the Completeness Theorem consists always of two parts. First we have show that *all formulas that have a proof are tautologies*. This implication is also called a Soundness Theorem, or soundness part of the Completeness Theorem. The second implication says: *if a formula is a tautology then it has a proof*. This alone is often called a Completeness Theorem. In our case, we call it a completeness part of the Completeness Theorem.

The proof of the soundness part is standard. We concentrate here on the completeness part of the Completeness Theorem and present two proofs of it.

The first proof is straightforward. It shows how one can use the assumption that a formula A is a tautology in order to construct its formal proof. It is hence called *a proof - construction method*.

The second proof shows how one can deduce that *a formula A is not a tautology from the fact that it does not have a proof*. It is hence called a *counter-model construction method*.

All these proofs and considerations are relative to a proof system whose completeness we discuss and its semantics.

The semantics is, of course, that for classical propositional logic, so when we write

$$\models A$$

we mean that A is a classical propositional tautology.

As far as the proof system is concerned we define here a certain class $\mathcal{S}$ of proof systems, instead of one proof system. We show that the Completeness Theorem holds for any system S from this class $\mathcal{S}$. In particular, our system H_2 from chapter 8 is complete, as it belongs to the class of systems $\mathcal{S}$.

1 Classical Propositional System H_2

There are many Hilbert style proof systems for the classical propositional calculus. We present here one of them as it was called defined in chapter 8, and prove the Completeness theorem for it.

H_2 is the following proof system:

$$H_2 = (\ \mathcal{L}_{\{\Rightarrow,\neg\}},\ \ \{A1, A2, A3\},\ \ MP\ ) \tag{1}$$

where $A1, A2, A3$ are axioms of the system defined below, MP is its rule of inference, called Modus Ponens is called a Hilbert proof system for the classical propositional logic. The axioms $A1 - A3$ are defined as follows.

A1 $(A \Rightarrow (B \Rightarrow A))$,

A2 $((A \Rightarrow (B \Rightarrow C)) \Rightarrow ((A \Rightarrow B) \Rightarrow (A \Rightarrow C)))$,

A3 $((\neg B \Rightarrow \neg A) \Rightarrow ((\neg B \Rightarrow A) \Rightarrow B)))$

MP (Rule of inference)

$$(MP)\ \frac{A\ ;\ (A \Rightarrow B)}{B},$$

and A, B, C are any formulas of the propositional language $\mathcal{L}_{\{\Rightarrow,\neg\}}$.

We write, as before

$$\vdash_{H_2} A$$

to denote that a formula A has a formal proof in H_2 (from the set of logical axioms $A1, A2, A3$), and

$$\Gamma \vdash_{H_2} A$$

to denote that a formula A has a formal proof in H_2 from a set of formulas Γ (and the set of logical axioms $A1, A2, A3$.

Obviously, the selected axioms $A1, A2, A3$ are tautologies, and the Modus Ponens rule leads from tautologies to tautologies, hence our proof system H_2 is *sound* i.e. the following theorem holds.

Theorem 1.1 (Soundness Theorem) *For every formula $A \in \mathcal{F}$,*

$$if\ \ \vdash_{H_2} A,\ \ then\ \ \models A.$$

The soundness theorem proves that our prove system "produces" only tautologies. We show, as the next step, that our proof system "produces" not only tautologies, but all of the tautologies. This is called a *completeness theorem*.

The proof of completeness theorem for a given semantics and a given proof system is always a main point in any logic creation. There are many ways (techniques) to prove it, depending on the proof system and on the given semantics.

We present here two proofs of the completeness theorem for our system H_2 as also defined in Chapter 8.

The first proof is presented in the section 3. It is very elegant and simple, but is only applicable to the classical propositional logic semantics and proof systems. It is, as the proof of Deduction Theorem, a fully constructive proof.

The technique it uses, because of its specifics can't even be used in a case of classical predicate logic, not to mention non-classical logics.

The second proof is presented the section 4. The techniques defined in this proof are as you will see are much more complicated. Their strength and importance lies in a fact that they can be applied in an extended version to the proof of completeness for classical predicate logic and many non-classical propositional and predicate logics.

The second proof is based on the fact that it provides a method of a construction of a counter-model for a formula A based on the knowledge that A is not provable. This means that one can prove that a formula A is not a tautology from the fact that it does not have a proof.

The way we define a counter-model for any non-provable A is much more general (and less constructive) then in the case of our first proof in section 3. We hence call it a *a counter-model existence method*.

The importance of this method lies, as we mentioned before, in the fact that it generalizes to the case of predicate logic, and many of non-classical logics; propositional and predicate. It is hence a much more general method then the first one and this is the reason we present it here.

2 The System S

In fact, the two proofs of Completeness Theorem can be performed for any proof system S for classical propositional logic in which the formulas 1, 3, 4, and 7-9 stated in lemma 4.1, Chapter 8 and all axioms of the system H_2 are provable. We assume provability of these formulas as they are formulas used in the proof of Deduction Theorem, and in both proofs of the Completeness Theorem.

It means that both proofs are valid for any proof system define as follows.

Let

$$S = (\ \mathcal{L}_{\{\Rightarrow,\neg\}},\quad AX,\quad MP\ )$$

be **a sound** proof system with a set of logical axioms AX such that all **formulas listed below are provable** in S.

$$(A \Rightarrow (B \Rightarrow A)), \tag{2}$$

$$((A \Rightarrow (B \Rightarrow C)) \Rightarrow ((A \Rightarrow B) \Rightarrow (A \Rightarrow C))), \tag{3}$$

$$((\neg B \Rightarrow \neg A) \Rightarrow ((\neg B \Rightarrow A) \Rightarrow B)), \tag{4}$$

$$(A \Rightarrow A), \tag{5}$$

$$(B \Rightarrow \neg\neg B), \tag{6}$$

$$(\neg A \Rightarrow (A \Rightarrow B)), \tag{7}$$

$$(A \Rightarrow (\neg B \Rightarrow \neg(A \Rightarrow B))), \tag{8}$$

$$((A \Rightarrow B) \Rightarrow ((\neg A \Rightarrow B) \Rightarrow B)), \tag{9}$$

$$((\neg A \Rightarrow A) \Rightarrow A), \tag{10}$$

We present here two proofs of the following theorem.

Theorem 2.1 (Completeness Theorem) *For any formula A of S,*

$$\models A \quad if \quad and \quad only \quad if \quad \vdash_S A.$$

OBSERVATION 1
The formulas (2) - (4) are axioms of H_2 and formulas (5) - (10) have proofs in it, by the lemma 4.1, Chapter 8 and we have proved that the system H_2 is sound, hence the Completeness Theorem for the system S implies the completeness of the system H_2. We get, as a particular case of the theorem 2.1 the following theorem.

Theorem 2.2 (Completeness Theorem for H_2) *For any formula A of H_2,*

$$\models A \quad if \quad and \quad only \quad if \quad \vdash_{H_2} A.$$

OBSERVATION 2
We have assumed that the system S is sound, i.e. that the following theorem holds for S.

Theorem 2.3 (Soundness Theorem)

For any formula A of S,

$$if \ \vdash_S A, \quad then \models A.$$

It means that in order to prove the Completeness Theorem 2.1 we need to prove only the following implication.

For any formula A of S,

$$If \ \models A, \ \ then \ \vdash_S A. \tag{11}$$

Both proofs of the Completeness Theorem relay on the Deduction Theorem, as discussed and proved in the previous chapter.

This theorem was proved for the system H_1 that is different that S, but note, that only the formulas (2), (3) and (5) were used in its proof, so the proof holds for the system S as well, as it held for the system H_2, i.e. we have the following theorem.

Theorem 2.4 (Deduction Theorem for S) *For any formulas A, B of S and Γ be any subset of formulas of S,*

$$\Gamma, \ A \vdash_S B \ \ if \ \ and \ \ only \ \ if \ \ \Gamma \vdash_S (A \Rightarrow B). \tag{12}$$

3 Proof 1: Proof Construction Method

The proof presented here is similar in its structure to the proof of the deduction theorem and is due to Kalmar, 1935. It is a *constructive proof*. It shows how one can use the assumption that a formula A is a tautology in order to construct its formal proof. We hence call it *a proof construction method*. It relies heavily on the Deduction Theorem.

In order to prove that any tautology has a formal proof in S, we need first to present one definition and to prove one lemma. We write $\vdash \ A$ instead of $\vdash_S A$, as the system S is fixed.

Definition 3.1 *Let A be a formula and $b_1, b_2, ..., b_n$ be all propositional variables that occur in A. Let v be variable assignment $v : VAR \longrightarrow \{T, F\}$. We define, for $A, b_1, b_2, ..., b_n$ and v a corresponding formulas A', $B_1, B_2, ..., B_n$ as follows:*

$$A' = \begin{cases} A & \text{if } v^*(A) = T \\ \neg A & \text{if } v^*(A) = F \end{cases}$$

$$B_i = \begin{cases} b_i & \text{if } v(b_i) = T \\ \neg b_i & \text{if } v(b_i) = F \end{cases}$$

for $i = 1, 2, ..., n$.

Example 1

Let A be a formula

$$(a \Rightarrow \neg b) \tag{13}$$

and let v be such that

$$v(a) = T, \quad v(b) = F. \tag{14}$$

In this case $b_1 = a,\ b_2 = b$, and $v^*(A) = v^*(a \Rightarrow \neg b) = v(a) \Rightarrow \neg v(b) = T \Rightarrow \neg F = T$. The corresponding A', B_1, B_2 are:

$A' = A$ (as $v^*(A) = T$),

$B_1 = a$ (as $v(a) = T$),

$B_2 = \neg b$ (as $v(b) = F$).

Example 2

Let A be a formula

$$((\neg a \Rightarrow \neg b) \Rightarrow c)$$

and let v be such that

$$v(a) = T, \quad v(b) = F, v(c) = F.$$

Evaluate A', $B_1, ...B_n$ as defined by the definition 3.1.

In this case $n = 3$ and $b_1 = a,\ b_2 = b, b_3 = c$, and $v^*(A) = v^*((\neg a \Rightarrow \neg b) \Rightarrow c) = ((\neg v(a) \Rightarrow \neg v(b)) \Rightarrow v(c)) = ((\neg T \Rightarrow \neg F) \Rightarrow F) = (T \Rightarrow F) = F$.

The corresponding A', B_1, B_2, B_2 are:

$A' = \neg A = \neg((\neg a \Rightarrow \neg b) \Rightarrow c)$ (as $v^*(A) = F$),

$B_1 = a$ (as $v(a) = T$),

$B_2 = \neg b$ (as $v(b) = F$).

$B_3 = \neg c$ (as $v(c) = F$).

The lemma stated below describes a method of transforming a semantic notion of a tautology into a syntactic notion of provability. It defines, for any formula A and a variable assignment v a corresponding deducibility relation $\vdash$.

Lemma 3.1 (Main Lemma) *For any formula A and a variable assignment v, if $A^{'}$, B_1 , B_2, ..., B_n are corresponding formulas defined by 3.1, then*

$$B_1, B_2, ..., B_n \ \vdash \ A'. \tag{15}$$

Example 3

Let A, v be as defined by 13, 14, then the lemma 3.1 asserts that

$$a, \neg b \ \vdash \ (a \Rightarrow \neg b).$$

Example 4

Let A, v be as defined in example 2, then the lemma 3.1 asserts that

$$a, \neg b, \neg c \ \vdash \ \neg((\neg a \Rightarrow \neg b) \Rightarrow c)$$

3.1 Proof of the Main Lemma

The Main Lemma 3.1 states:

For any formula A and a variable assignment v, if $A^{'}$, B_1 , B_2, ..., B_n are corresponding formulas defined by definition 3.1, then

$$B_1, B_2, ..., B_n \ \vdash \ A'.$$

The proof is by induction on the degree of A i.e. a number n of logical connectives in A.

Case: $n = 0$

In the case that $n = 0$ A is atomic and so consists of a single propositional variable, say a. We have to cases to consider, $v^*(A) = T$ or $v^*(A) = F$. Clearly, if $v^*(A) = T$ then we $A' = A = a$, $B_1 = a$, and $a \vdash a$ holds by the Deduction Theorem and (5). I.e. $\vdash (a \ \Rightarrow \ a)$ holds by (5) and applying the the Deduction Theorem we get $a \vdash a$.

If $v^*(A) = F$ then we $A' = \neg A = \neg a$, $B_1 = \neg a$, and $\vdash (\neg a \ \Rightarrow \ \neg a)$ holds by (5). Applying the the Deduction Theorem we get $\neg a \vdash \neg a$. So the lemma holds for the case $n = 0$.

Now assume that the lemma holds for any A with $j < n$ logical connectives (any A of the degree $j < n$). The goal is to prove that it holds for A with the degree n.
There are several subcases to deal with.

Case: A **is** $\neg A_1$

If A is of the form $\neg A_1$ then A_1 has less then n connectives and by the inductive assumption we have the formulas $A_1^{'}$, B_1 , B_2, ..., B_n corresponding to the A_1 and the propositional variables $b_1, b_2, ..., b_n$ in A_1, as defined by the definition 3.1, such that

$$B_1, B_2, ..., B_n \ \vdash \ A_1^{'}. \tag{16}$$

Observe, that the formulas A and $\neg A_1$ have the same propositional variables, so the corresponding formulas B_1 , B_2, ..., B_n are the same for both of them. We are going to show that the inductive assumption (16) allows us to prove that the lemma holds for A, ie. that

$$B_1, B_2, ..., B_n \ \vdash \ A^{'}.$$

There two cases to consider.

Case: $v^*(A_1) = T$

If $v^*(A_1) = T$ then by definition 3.1 $A_1^{'} = A_1$ and by the inductive assumption (16)

$$B_1, B_2, ..., B_n \ \vdash \ A_1. \tag{17}$$

In this case $v^*(A) = v^*(\neg A_1) = \neg v^*(T) = F$ and so $A^{'} = \neg A = \neg\neg A_1$. Since we have assumed (6), i.e. that $\vdash \ (A_1 \Rightarrow \neg\neg A_1)$ we have by the monotonicity that also $B_1, B_2, ..., B_n \ \vdash \ (A_1 \Rightarrow \neg\neg A_1)$. By (17) and Modus Ponens also $B_1, B_2, ..., B_n \ \vdash \ \neg\neg A_1$, that is $B_1, B_2, ..., B_n \ \vdash \ \neg A$, that is $B_1, B_2, ..., B_n \ \vdash \ A^{'}$.

Case: $v^*(A_1) = F$

If $v^*(A_1) = F$ then $A_1^{'} = \neg A_1$ and $v^*(A) = T$ so $A^{'} = A$. Therefore the inductive assumption (16) $B_1, B_2, ..., B_n \ \vdash \ \neg A_1$, that is $B_1, B_2, ..., B_n \ \vdash \ A^{'}$.

Case: A **is** $(A_1 \Rightarrow A_2)$

$A = A(b_1, ... b_n)$ so there are some subsequences $c_1, ..., c_k$ and $d_1, ...d_m$ ($k, m \leq n$) of the sequence $b_1, ..., b_n$ such that $A_1 = A_1(c_1, ..., c_k)$ and $A_2 = A(d_1, ...d_m)$. A_1 and A_2 have less than n connectives and so by the inductive assumption we have appropriate formulas $C_1, ..., C_k$ and $D_1, ...D_m$ such that

$$C_1, C_2, ..., C_k \ \vdash \ A_1{}^{'} \ \text{ and } \ D_1, D_2, ..., D_m \ \vdash \ A_2{}^{'}$$

and $C_1, C_2, ..., C_k, D_1, D_2, ..., D_m$ are subsequences of formulas $B_1, B_2, ..., B_n$ corresponding to the propositional variables in A. By monotonicity we have

$$B_1, B_2, ..., B_n \ \vdash \ A_1{}^{'} \text{ and } B_1, B_2, ..., B_n \ \vdash \ A_2{}^{'}. \tag{18}$$

Here we have the following subcases to consider.

Case: $v^*(A_1) = v^*(A_2) = T$

If $v^*(A_1) = T$ then $A_1{}'$ is A_1 and if $v^*(A_2) = T$ then $A_2{}'$ is A_2. We also have $v^*(A_1 \Rightarrow A_2) = T$ and so A' is $(A_1 \Rightarrow A_2)$. By the above and ( ) we have that $B_1, B_2, ..., B_n \vdash A_2$ and since we have assumed (2) i.e. $\vdash (A_2 \Rightarrow (A_1 \Rightarrow A_2))$, we have by monotonicity and Modus Ponens, that $B_1, B_2, ..., B_n \vdash (A_1 \Rightarrow A_2)$, that is $B_1, B_2, ..., B_n \vdash A'$.

Case: $v^*(A_1) = T, v^*(A_2) = F$

If $v^*(A_1) = T$ then $A_1{}'$ is A_1 and if $v^*(A_2) = F$ then $A_2{}'$ is $\neg A_2$. Also we have in this case $v^*(A_1 \Rightarrow A_2) = F$ and so A' is $\neg(A_1 \Rightarrow A_2)$. By the above and (3.1) we have that $B_1, B_2, ..., B_n \vdash A_1$ and $B_1, B_2, ..., B_n \vdash \neg A_2$. Since we have assumed formula (8), i.e. $\vdash (A_1 \Rightarrow (\neg A_2 \Rightarrow \neg(A_1 \Rightarrow A_2)))$, we have by monotonicity and Modus Ponens twice, that $B_1, B_2, ..., B_n \vdash \neg(A_1 \Rightarrow A_2)$, that is $B_1, B_2, ..., B_n \vdash A'$.

Case: $v^*(A_1) = F$

If $v^*(A_1) = F$ then $A_1{}'$ is $\neg A_1$ and, whatever value v gives A_2, we have $v^*(A_1 \Rightarrow A_2) = T$ and so A' is $(A_1 \Rightarrow A_2)$. Therefore, by (3.1) and above $B_1, B_2, ..., B_n \vdash \neg A_1$ and since by formula (7) we have $\vdash (\neg A_1 \Rightarrow (A_1 \Rightarrow A_2))$. By monotonicity and Modus Ponens we get that $B_1, B_2, ..., B_n \vdash (A_1 \Rightarrow A_2)$, that is $B_1, B_2, ..., B_n \vdash A'$.

With that we have covered all cases and, by induction on n, the proof of the lemma is complete.

3.2 Proof 1 of the Completeness Theorem

Now we use the Main Lemma 3.1 to prove the Completeness Theorem 2.1 i.e. to prove the following implication: for any formula A of S,

$$if \quad \models A \quad then \quad \vdash A.$$

Assume that $\models A$. Let $b_1, b_2, ..., b_n$ be all propositional variables that occur in A, i.e. $A = A(b_1, b_2, ..., b_n)$.

Let $v : VAR \to \{T, F\}$ be any variable assignment, and

$$v_A : \quad \{b_1, b_2, ..., b_n\} \quad \to \{T, F\} \tag{19}$$

its restriction to the formula A, i.e. $v_A = v|\{b_1, b_2, ..., b_n\}$. Let

$$V_A = \{v_A : \quad v_A : \{b_1, b_2, ..., b_n\} \quad \to \{T, F\}\} \tag{20}$$

By the Main Lemma 3.1 and the assumption that $\models A$ any $v \in V_A$ defines formulas B_1 , B_2, ..., B_n such that

$$B_1, B_2, ..., B_n \ \vdash \ A. \tag{21}$$

The proof is based on a method of using all $v \in V_A$ to define a process of elimination of all hypothesis $B_1, B_2, ..., B_n$ in 21) to finally construct the proof of A in S i.e. to prove that $\vdash \ A$.

Step 1: elimination of B_n.

Observe that by definition 3.1, each B_i is b_i or $\neg b_i$ depending on the choice of $v \in V_A$. In particular $B_n = b_n$ or $B_n = \neg b_n$. We choose two truth assignments $v_1 \neq v_2 \in V_A$ such that

$$v_1|\{b_1, ..., b_{n-1}\} = v_2|\{b_1, ..., b_{n-1}\} \tag{22}$$

and $v_1(b_n) = T$ and $v_2(b_n) = F$.

Case 1: $v_1(b_n) = T$, by definition 3.1 $B_n = b_n$. By the property (22), assumption that $\models A$, and the Main Lemma 3.1 applied to v_1

$$B_1, B_2, ..., B_{n-1}, b_n \ \vdash \ A.$$

By Deduction Theorem 2.4 we have that

$$B_1, B_2, ..., B_{n-1} \ \vdash \ (b_n \Rightarrow A). \tag{23}$$

Case 2: $v_2(b_n) = F$ hence by definition 3.1 $B_n = \neg b_n$. By the property (22), assumption that $\models A$, and the Main Lemma 3.1 applied to v_2

$$B_1, B_2, ...B_{n-1}, \neg b_n \ \vdash \ A.$$

By the Deduction Theorem 2.4 we have that

$$B_1, B_2, ..., B_{n-1} \ \vdash \ (\neg b_n \Rightarrow A). \tag{24}$$

By the assumed provability of the formula (9) for $A = b_n, B = A$ we have that

$$\vdash ((b_n \Rightarrow A) \Rightarrow ((\neg b_n \Rightarrow A) \Rightarrow A)).$$

By monotonicity we have that

$$B_1, B_2, ..., B_{n-1} \ \vdash \ ((b_n \Rightarrow A) \Rightarrow ((\neg b_n \Rightarrow A) \Rightarrow A)). \tag{25}$$

Applying Modus Ponens twice to the above property (25) and properties (23), (24) we get that

$$B_1, B_2, ..., B_{n-1} \ \vdash \ A. \tag{26}$$

and hence we have eliminated B_n.

Step 2: elimination of B_{n-1} from (26). We repeat the Step 1.

As before we have 2 cases to consider: $B_{n-1} = b_{n-1}$ or $B_{n-1} = \neg b_{n-1}$. We choose two truth assignments $w_1 \neq w_2 \in V_A$ such that

$$w_1|\{b_1, ..., b_{n-2}\} = w_2|\{b_1, ..., b_{n-2}\} = v_1|\{b_1, ..., b_{n-2}\} = v_2|\{b_1, ..., b_{n-2}\} \tag{27}$$

and $w_1(b_{n-1}) = T$ and $w_2(b_{n-1}) = F$.

As before we apply Main Lemma, Deduction Theorem, monotonicity, proper substitutions of assumed provability of the formula (9)i.e the fact that $\vdash \ ((A \Rightarrow B) \Rightarrow ((\neg A \Rightarrow B) \Rightarrow B))$, and Modus Ponens twice and eliminate B_{n-1} just as we eliminated B_n.

After n steps, we finally obtain that

$$\vdash \ A.$$

This ends the proof of Completeness Theorem.

Observe that our proof of the fact that $\vdash A$ is a constructive one. Moreover, we have used in it only Main Lemma 3.1 and Deduction Theorem 2.4 which both have a constructive proofs. We can hence reconstruct proofs in each case when we apply these theorems back to the original axioms of the system S, and in particular to the oroginal axioms $A1 - A3$ of H_2. The same applies to the proofs in H_2 of all formulas (2) - (10) of the system S. It means that for any A, such that $\models \ A$, the set V_A of all v restricted to A provides us a method of a construction of the formal proof of A in H_2, or in any system S in which formulas (2) - (10) are provable.

3.3 Proof 1: Examples and Exercises

Example 3.1 *As an example of how the proof of the Completeness Theorem works, we consider the case in which A is a following tautology*

$$(a \Rightarrow (\neg a \Rightarrow b))$$

and show how we use them to show that $\vdash A$.

We apply the Main Lemma 3.1 to all possible variable assignments $v \in V_A$. We have 4 variable assignments to consider.

Case 1: $v(a) = T, v(b) = T$.

In this case $B_1 = a, B_2 = b$ and, as in all cases $A' = A$ and by the lemma 3.1

$$a, b \ \vdash \ (a \Rightarrow (\neg a \Rightarrow b)).$$

Case 2: $v(a) = T, v(b) = F$.
In this case $B_1 = a, B_2 = \neg b$ and by the lemma 3.1

$$a, \neg b \ \vdash \ (a \Rightarrow (\neg a \Rightarrow b)).$$

Case 3: $v(a) = F, v(b) = T$.
In this case $B_1 = \neg a, B_2 = b$ and by the lemma 3.1

$$\neg a, b \ \vdash \ (a \Rightarrow (\neg a \Rightarrow b)).$$

Case 4: $v(a) = F, v(b) = F$.
In this case $B_1 = \neg a, B_2 = \neg b$ and by the lemma 3.1

$$\neg a, \neg b \ \vdash \ (a \Rightarrow (\neg a \Rightarrow b)).$$

Applying the Deduction Theorem to the cases above we have that

D1 (Cases 1 and 2)

$$a \ \vdash \ (b \Rightarrow (a \Rightarrow (\neg a \Rightarrow b))),$$
$$a \ \vdash \ (\neg b \Rightarrow (a \Rightarrow (\neg a \Rightarrow b))),$$

D2 (Cases 2 and 3)

$$\neg a \ \vdash \ (b \Rightarrow (a \Rightarrow (\neg a \Rightarrow b))),$$
$$\neg a \ \vdash \ (\neg b \Rightarrow (a \Rightarrow (\neg a \Rightarrow b))).$$

By the monotonicity and formula (9) we have that

$$a \ \vdash \ ((b \Rightarrow (a \Rightarrow (\neg a \Rightarrow b))) \Rightarrow ((\neg b \Rightarrow (a \Rightarrow (\neg a \Rightarrow b))) \Rightarrow (a \Rightarrow (\neg a \Rightarrow b))),$$
$$\neg a \ \vdash \ ((b \Rightarrow (a \Rightarrow (\neg a \Rightarrow b))) \Rightarrow ((\neg b \Rightarrow (a \Rightarrow (\neg a \Rightarrow b))) \Rightarrow (a \Rightarrow (\neg a \Rightarrow b))).$$

Applying Modus Ponens twice to **D1**, **D2** and these above, respectively, gives us

$$a \ \vdash \ (a \Rightarrow (\neg a \Rightarrow b)) \ and$$
$$\neg a \ \vdash \ (a \Rightarrow (\neg a \Rightarrow b)).$$

Applying the Deduction Theorem to the above we obtain

D3 $\vdash \ (a \Rightarrow (a \Rightarrow (\neg a \Rightarrow b)))$ and

D4 $\vdash \ (\neg a \Rightarrow (a \Rightarrow (\neg a \Rightarrow b)))$.

Applying Modus Ponens twice to **D3** and **D4** and the following form of 9, $\vdash \ ((a \Rightarrow (a \Rightarrow (\neg a \Rightarrow b))) \Rightarrow ((\neg a \Rightarrow (a \Rightarrow (\neg a \Rightarrow b))) \Rightarrow (a \Rightarrow (\neg a \Rightarrow b))))$ we get finally $(a \Rightarrow (\neg a \Rightarrow b))$ is provable in S, i.e. we have proved that

$$\vdash \ (a \Rightarrow (\neg a \Rightarrow b)).$$

Example 3.2 *The proof of Completeness Theorem defines a method of efficiently combining $v \in V_A$ while constructing the proof of A. Let consider the following tautology $A = A(a, b, c)$*

$$((\neg a \Rightarrow b) \Rightarrow (\neg(\neg a \Rightarrow b) \Rightarrow c).$$

We present bellow all steps of Proof 1 as applied to A.

By the Main Lemma 3.1 and the assumption that $\models A(a, b, c)$ any $v \in V_A$ defines formulas B_a , B_b, B_c such that

$$B_a, B_b, B_c \vdash A. \tag{28}$$

The proof is based on a method of using all $v \in V_A$ (there is 16 of them) to define a process of elimination of all hypothesis B_a, B_b, B_c in (28) to construct the proof of A in S i.e. to prove that $\vdash A$.

Step 1: elimination of B_c.

Observe that by definition 3.1, B_c is c or $\neg c$ depending on the choice of $v \in V_A$. We choose two truth assignments $v_1 \neq v_2 \in V_A$ such that

$$v_1|\{a, b\} = v_2|\{a, b\} \tag{29}$$

and $v_1(c) = T$ and $v_2(c) = F$.

Case 1: $v_1(c) = T$, by definition 3.1 $B_c = c$. By the property (29), assumption that $\models A$, and the Main Lemma 3.1 applied to v_1

$$B_a, B_b, c \vdash A.$$

By Deduction Theorem 2.4 we have that

$$B_a, B_b \vdash (c \Rightarrow A). \tag{30}$$

Case 2: $v_2(c) = F$ hence by definition 3.1 $B_c = \neg c$. By the property (29), assumption that $\models A$, and the Main Lemma 3.1 applied to v_2

$$B_a, B_b, \neg c \vdash A.$$

By the Deduction Theorem 2.4 we have that

$$B_a, B_b \vdash (\neg c \Rightarrow A). \tag{31}$$

By the assumed provability of the formula (9) for $A = c, B = A$ we have that

$$\vdash ((c \Rightarrow A) \Rightarrow ((\neg c \Rightarrow A) \Rightarrow A)).$$

By monotonicity we have that

$$B_a, B_b \ \vdash \ ((c \Rightarrow A) \Rightarrow ((\neg c \Rightarrow A) \Rightarrow A)). \tag{32}$$

Applying Modus Ponens twice to the above property (32) and properties (30), (31) we get that

$$B_a, B_b \ \vdash \ A. \tag{33}$$

and hence we have eliminated B_c.

Step 2: elimination of B_b from (33). We repeat the Step 1.
As before we have 2 cases to consider: $B_b = b$ or $B_b = \neg b$. We choose from V_A two truth assignments $w_1 \neq w_2 \in V_A$ such that

$$w_1|\{a\} = w_2|\{a\} = v_1|\{a\} = v_2|\{a\} \tag{34}$$

and $w_1(b) = T$ and $w_2(b) = F$.

Case 1: $w_1(b) = T$, by definition 3.1 $B_b = b$. By the property (34), assumption that $\models A$, and the Main Lemma 3.1 applied to w_1

$$B_a, b \ \vdash \ A.$$

By Deduction Theorem 2.4 we have that

$$B_a \ \vdash \ (b \Rightarrow A). \tag{35}$$

Case 2: $w_2(c) = F$ hence by definition 3.1 $B_b = \neg b$. By the property (34), assumption that $\models A$, and the Main Lemma 3.1 applied to w_2

$$B_a, \neg b \ \vdash \ A.$$

By the Deduction Theorem 2.4 we have that

$$B_a \ \vdash \ (\neg b \Rightarrow A). \tag{36}$$

By the assumed provability of the formula (9) for $A = b, B = A$ we have that

$$\vdash ((b \Rightarrow A) \Rightarrow ((\neg b \Rightarrow A) \Rightarrow A)).$$

By monotonicity we have that

$$B_a \ \vdash \ ((b \Rightarrow A) \Rightarrow ((\neg b \Rightarrow A) \Rightarrow A)). \tag{37}$$

Applying Modus Ponens twice to the above property (37) and properties (35), (36) we get that

$$B_a \vdash A. \tag{38}$$

and hence we have eliminated B_b.

Step 3: elimination of B_a from (38). We repeat the Step 2.
As before we have 2 cases to consider: $B_a = a$ or $B_a = \neg a$. We choose from V_A two truth assignments $g_1 \neq g_2 \in V_A$ such that

$$g_1(a) = T \text{ and } g_2(a) = F. \tag{39}$$

Case 1: $g_1(a) = T$, by definition 3.1 $B_a = a$. By the property (39), assumption that $\models A$, and the Main Lemma 3.1 applied to g_1

$$a \vdash A.$$

By Deduction Theorem 2.4 we have that

$$\vdash (a \Rightarrow A). \tag{40}$$

Case 2: $g_2(a) = F$ hence by definition 3.1 $B_a = \neg a$. By the property (39), assumption that $\models A$, and the Main Lemma 3.1 applied to g_2

$$\neg a \vdash A.$$

By the Deduction Theorem 2.4 we have that

$$\vdash (\neg a \Rightarrow A). \tag{41}$$

By the assumed provability of the formula (9) for $A = a, B = A$ we have that

$$\vdash ((a \Rightarrow A) \Rightarrow ((\neg a \Rightarrow A) \Rightarrow A)). \tag{42}$$

Applying Modus Ponens twice to the above property (42) and properties (40), (41) we get that

$$\vdash A. \tag{43}$$

and hence we have eliminated B_a, B_b and B_c.

Problem 1

For the formulas A_i and corresponding truth assignments v find formulas $B_1, ..B_k, A_i^{'}$ as described by the Main Lemma 3.1, i.e. such that

$$B_1, ...B_k \vdash A_i^{'}.$$

1. $A_1 = ((\neg(b \Rightarrow a) \Rightarrow \neg a) \Rightarrow ((\neg b \Rightarrow (a \Rightarrow \neg c)) \Rightarrow c))$
$v(a) = T, v(b) = F, v(c) = T.$

2. $A_2 = ((a \Rightarrow (c \Rightarrow (\neg b \Rightarrow c))) \Rightarrow ((\neg d \Rightarrow (a \Rightarrow (\neg a \Rightarrow b))) \Rightarrow (a \Rightarrow (\neg a \Rightarrow b))))$
$v(a) = F, v(b) = F, v(c) = T, v(d) = F$

3. $A_3 = (\neg b \Rightarrow (c \Rightarrow (\neg a \Rightarrow b)))$
$v(a) = F, v(b) = F, v(c) = T$

4. $A_4 = (\neg a_1 \Rightarrow (a_2 \Rightarrow (\neg a_3 \Rightarrow a_1)))$
$v(a_1) = F, v(a_2) = F, v(a_3) = T$

5. $A_5 = ((b \Rightarrow (a_1 \Rightarrow (\neg c \Rightarrow b))) \Rightarrow ((\neg b \Rightarrow (a_2 \Rightarrow (\neg a_1 \Rightarrow b))) \Rightarrow (c \Rightarrow (\neg a \Rightarrow b)))$
$v(a) = F, v(b) = T, v(c) = F, v(a_1) = T, v(a_2) = F$

Problem 2 For any of the formulas A_1, A_2, A_3, A_4 listed below construct their proofs, as described in the proof of the Completeness Theorem. Follow example 3.1, or example 3.2.

$A_1 = (\neg\neg b \Rightarrow b)$

$A_2 = ((a \Rightarrow b) \Rightarrow (\neg b \Rightarrow \neg a))$

$A_3 = (\neg(a \Rightarrow b) \Rightarrow \neg(\neg b \Rightarrow \neg a))$

$A_4 = (\neg(\neg(a \Rightarrow \neg b) \Rightarrow \neg c) \Rightarrow \neg(b \Rightarrow \neg c))$

$A_5 = ((a \Rightarrow (b \Rightarrow \neg a)) \Rightarrow (\neg(b \Rightarrow \neg a) \Rightarrow \neg a)).$

4 Proof 2: A Counter- Model Existence Method

We prove now the Completeness Theorem 2.1 by proving the opposite implication:

$$if \quad \nvdash A, \ then \ \not\models A \tag{44}$$

instead of the implication 11.

We will show now how one can define of a counter-model for A from the fact that A is not provable. This means that we deduce that a formula A is not

a tautology from the fact that it does not have a proof. We hence call it a *a counter-model existence method.*

The definition of the counter-model for any non-provable A is much more general (and less constructive) then in the case of our first proof in section 3. It can be generalized to the case of predicate logic, and many of non-classical logics; propositional and predicate. It is hence a much more general method then the first one and this is the reason we present it here.

We remind that $\not\models A$ means that there is a variable assignment $v : VAR \longrightarrow \{T, F\}$, such that $v^*(A) \neq T$, i.e. in classical semantics that $v^*(A) = F$. a Such v is called a counter-model for A, hence the proof provides a counter-model construction method.

Since we assume in 44 that A does not have a proof in S ($\nvdash A$) the method uses this information in order to show that A is not a tautology, i.e. to define v such that $v^*(A) = F$. We also have to prove that all steps in that method are correct. This is done in the following steps.

Step 1: Definition of Δ^*

We use the information $\nvdash A$ to define a special set Δ^*, such that $\neg A \in \Delta^*$.

Step 2: Counter - model definition

We define the variable assignment $v : VAR \longrightarrow \{T, F\}$ as follows:

$$v(a) = \begin{cases} T & \text{if } \Delta^* \vdash a \\ F & \text{if } \Delta^* \vdash \neg a. \end{cases}$$

Step 3: Prove that v is a counter-model

We first prove a more general property, namely we prove that the set Δ^* and v defined in the steps 1 and 2, respectively, are such that for every formula $B \in \mathcal{F}$,

$$v^*(B) = \begin{cases} T & \text{if } \Delta^* \vdash B \\ F & \text{if } \Delta^* \vdash \neg B. \end{cases}$$

Then we use the **Step 1** to prove that $v^*(A) = F$.

The definition and the properties of the set Δ^*, and hence the **Step 1**, are the most essential for the proof. The other steps have only technical character. The main notions involved in this step are: *consistent set, complete set* and a *consistent complete extension of a set.* We are going now to introduce them and to prove some essential facts about them.

4.0.1 Consistent and Inconsistent Sets

There exist two definitions of consistency; semantical and syntactical. The **semantical** uses definition the notion of a model and says:

a set is consistent if it has a model.

The **syntactical** one uses the notion of provability and says:

a set is consistent if one can't prove a contradiction from it.

In our proof of the Completeness Theorem we use assumption that a given formula A does not have a proof to deduce that A is not a tautology. We hence use the following syntactical definition of consistency.

Consistent set

We say that a set $\Delta \subseteq \mathcal{F}$ of formulas is consistent if and only if **there is no** a formula $A \in \mathcal{F}$ such that

$$\Delta \vdash A \quad \text{and} \quad \Delta \vdash \neg A. \tag{45}$$

Inconsistent set

A set $\Delta \subseteq \mathcal{F}$ is inconsistent if and only if **there is** a formula $A \in \mathcal{F}$ such that $\Delta \vdash A$ and $\Delta \vdash \neg A$.

The notion of consistency, as defined above, is characterized by the following lemma.

Lemma 4.1 (Consistency Condition)

For every set $\Delta \subseteq \mathcal{F}$ of formulas, the following conditions are equivalent:

(i) Δ *is* consistent*,*

(ii) *there is a formula $A \in \mathcal{F}$ such that $\Delta \nvdash A$.*

Proof The implications: **(i)** implies **(ii)** and vice-versa are proved by showing the corresponding opposite implications. I.e. to establish the equivalence of **(i)** and **(ii)**, we first show that **not (ii)** implies **not (i)**, and then that **not (i)** implies **not (ii)**.

Case 1

Assume that **not (ii)**. It means that for all formulas $A \in \mathcal{F}$ we have that $\Delta \vdash A$. In particular it is true for a certain $A = B$ and $A = \neg B$ and hence proves that Δ is inconsistent, i.e. **not (i)** holds.

Case 2

Assume that **not (i)**, i.e that Δ is inconsistent. Then there is a formula A such that $\Delta \vdash A$ and $\Delta \vdash \neg A$. Let B be any formula. Since $(\neg A \Rightarrow (A \Rightarrow B))$ is provable in S (formula 7) , hence by applying Modus Ponens twice and by detaching from it $\neg A$ first, and A next, we obtain a formal proof of B from the set Δ, so that $\Delta \vdash B$ for any formula B. Thus **not (ii)**.

The inconsistent sets are hence characterized by the following fact.

Lemma 4.2 (Inconsistency Condition)

For every set $\Delta \subseteq \mathcal{F}$ of formulas, the following conditions are equivalent:

(i) Δ *is* inconsistent*,*

(ii) *for all formulas $A \in \mathcal{F}$, $\Delta \vdash A$.*

We remind here the property of the finiteness of the consequence operation.

Lemma 4.3

For every set Δ of formulas and for every formula $A \in \mathcal{F}$, $\Delta \vdash A$ if and only if there is a finite subset $\Delta_0 \subseteq \Delta$ such that $\Delta_0 \vdash A$.

Proof If $\Delta_0 \vdash A$ for a certain $\Delta_0 \subseteq \Delta$, then by the monotonicity of the consequence, also $\Delta \vdash A$. Assume now that $\Delta \vdash A$ and let $A_1, A_2, ..., A_n$ be a formal proof of A from Δ. Let $\Delta_0 = \{A_1, A_2, ..., A_n\} \cap \Delta$. Obviously, Δ_0 is finite and $A_1, A_2, ..., A_n$ is a formal proof of A from Δ_0.

The following theorem is a simply corollary of the above lemma 4.3.

Theorem 4.1 (Finite Inconsistency)

If a set Δ is inconsistent, then there is a finite subset $\Delta_0 \subseteq \Delta$ which is inconsistent. It follows therefore from that if every finite subset of a set Δ is consistent, then the set Δ is also consistent.

Proof If Δ is inconsistent, then for some formula A, $\Delta \vdash A$ and $\Delta \vdash \neg A$. By above lemma 4.3, there are finite subsets Δ_1 and Δ_2 of Δ such that $\Delta_1 \vdash A$ and $\Delta_2 \vdash \neg A$. By monotonicity, the union $\Delta_1 \cup \Delta_2$ is a finite subset of Δ, such that $\Delta_1 \cup \Delta_2 \vdash A$ and $\Delta_1 \cup \Delta_2 \vdash \neg A$. Hence $\Delta_1 \cup \Delta_2$ is a finite inconsistent subset of Δ. The second implication is the opposite to the one just proved and hence also holds.

The following lemma links the notion of non-provability and consistency. It will be used as an important step in our proof of the Completeness Theorem.

Lemma 4.4

For any formula $A \in \mathcal{F}$, if $\nvdash A$, then the set $\{\neg A\}$ is consistent.

Proof If $\{\neg A\}$ is inconsistent, then by the Inconsistency Condition 4.2 we have $\{\neg A\} \vdash A$. This and the Deduction Theorem 12 imply $\vdash (\neg A \Rightarrow A)$. Applying the Modus Ponens rule to $\vdash (\neg A \Rightarrow A)$ and assumed provable formula 10 $((\neg A \Rightarrow A) \Rightarrow A)$, we get that $\vdash A$, contrary to the assumption of the lemma.

4.0.2 Complete and Incomplete Sets

Another important notion, is that of *a complete* set of formulas. Complete sets, as defined here are sometimes called *maximal*, but we use the first name for them.

They are defined as follows.

Complete set

A set Δ of formulas is called complete if **for every** formula $A \in \mathcal{F}$,

$$\Delta \vdash A \quad or \quad \Delta \vdash \neg A. \tag{46}$$

The complete sets are characterized by the following fact.

Lemma 4.5 (Complete set condition)

For every set $\Delta \subseteq \mathcal{F}$ of formulas, the following conditions are equivalent:

(i) Δ *is* complete*,*

(ii) *for every formula $A \in \mathcal{F}$, if $\Delta \nvdash A$, then the set $\Delta \cup \{A\}$ is inconsistent.*

Proof We consider two cases. We show that **(i)** implies **(ii)** and vice-versa, that **(ii)** also implies **(i)**.

Case 1

Assume that **(i)** and that for every formula $A \in \mathcal{F}$, $\Delta \nvdash A$, we have to show that in this case $\Delta \cup \{A\}$ is inconsistent. But if $\Delta \nvdash A$, then from the definition of complete set and assumption that Δ is complete set, we get that $\Delta \vdash \neg A$. By the monotonicity of the consequence we have that $\Delta \cup \{A\} \vdash \neg A$ as well. Since, by formula 5 we have $\vdash (A \Rightarrow A)$, by monotonicity $\Delta \vdash (A \Rightarrow A)$ and by Deduction Theorem $\Delta \cup \{A\} \vdash A$. This proves that $\Delta \cup \{A\}$ is inconsistent. Hence **(ii)** holds.

Case 2

Assume that **(ii)**. Let A be any formula. We want to show that the condition: $\Delta \vdash A$ or $\Delta \vdash \neg A$ is satisfied. If $\Delta \vdash \neg A$, then the condition is obviously satisfied.

If, on other hand, $\Delta \nvdash \neg A$, then we are going to show now that it must be , under the assumption of **(ii)**, that $\Delta \vdash A$, i.e. that **(i)** holds.

Assume that $\Delta \nvdash \neg A$, then by **(ii)**, the set $\Delta \cup \{\neg A\}$ is inconsistent. It means, by the Consistency Condition 4.1, that $\Delta \cup \{\neg A\} \vdash A$. By the Deduction Theorem 12, this implies that $\Delta \vdash (\neg A \Rightarrow A)$. Since $((\neg A \Rightarrow A) \Rightarrow A)$ is assumed to be provable in S (formula 4), by monotonicity $\Delta \vdash ((\neg A \Rightarrow A) \Rightarrow A)$. Detaching $(\neg A \Rightarrow A)$, we obtain that $\Delta \vdash A$, what ends the proof that **(i)** holds.

Incomplete set

A set Δ of formulas is called incomplete if it is not complete, i.e. if **there exists** a formula $A \in \mathcal{F}$ such that

$$\Delta \nvdash A \quad \textit{and} \quad \Delta \nvdash \neg A. \tag{47}$$

We get as a direct consequence of the lemma 4.5 the following characterization of incomplete sets.

Lemma 4.6 (Incomplete Set Condition)

For every set $\Delta \subseteq \mathcal{F}$ of formulas, the following conditions are equivalent:

(i) Δ *is* incomplete*,*

(ii) *there is formula $A \in \mathcal{F}$ such that $\Delta \nvdash A$ and the set $\Delta \cup \{A\}$ is consistent.*

4.0.3 Main Lemma: Complete Consistent Extension

Now we are going to prove a lemma that is essential to the construction of the special set Δ^* mentioned in the **Step 1** of the proof of the Completeness Theorem, and hence to the proof of the theorem itself. Let's first introduce one more notion.

> A set Δ^* of formulas is called an **extension** of a set Δ of formulas if the following condition holds.
>
> $$\{A \in \mathcal{F} : \ \Delta \ \vdash \ A\} \subseteq \{A \in \mathcal{F} : \ \Delta^* \ \vdash \ A\}.$$
>
> In this case we say also that Δ **extends** to the set of formulas Δ^*.

The Main Lemma states as follows.

Lemma 4.7 (Complete Consistent Extension)

Every consistent set Δ of formulas can be extended to a complete consistent set Δ^* of formulas.

Proof Assume that the lemma does not hold, i.e. that there is a consistent set Δ, such that all its consistent extensions are not complete. In particular, as Δ is an consistent extension of itself, we have that Δ is not complete.

The proof consists of a construction of a particular set Δ^* and proving that it forms a complete consistent extension of Δ, contrary to the assumption that all its consistent extensions are not complete.

Construction of Δ^*.

As we know, the set $\mathcal{F}$ of all formulas is enumerable. They can hence be put in an infinite sequence

$$A_1, A_2, ..., A_n, ... \tag{48}$$

such that every formula of $\mathcal{F}$ occurs in that sequence exactly once.

We define now, as the first step in the construction of Δ^*, an infinite sequence $\{\Delta_n\}_{n\in N}$ of consistent subsets of formulas together with a sequence $\{B\}_{n\in N}$ of formulas as follows.

Initial Step

In this step we define the sets Δ_1, Δ_2 and the formula B_1. We prove that Δ_1 and Δ_2 are consistent, incomplete extensions of Δ.

We take, as the first set, the set Δ, i.e. we define

$$\Delta_1 = \Delta. \tag{49}$$

Since, by assumption, the set Δ, and hence also Δ_1 is not complete, it follows from the Incomplete Set Condition 4.6, that there is a formula $B \in \mathcal{F}$ such that $\Delta_1 \nvdash B$, then and the set $\Delta_1 \cup \{B\}$ is consistent.

Let

$$B_1$$

be the first formula with this property in the sequence 48 of all formulas; we then define

$$\Delta_2 = \Delta_1 \cup \{B_1\}. \tag{50}$$

The set Δ_2 is consistent and $\Delta_1 = \Delta \subseteq \Delta_2$, so by the monotonicity, Δ_2 is a consistent extension of Δ. Hence Δ_2 cannot be complete.

Inductive Step

Suppose that we have defined a sequence

$$\Delta_1, \Delta_2, ..., \Delta_n$$

of incomplete, consistent extensions of Δ, and a sequence

$$B_1, B_2, ...B_{n-1}$$

of formulas, for $n \geq 2$.

Since Δ_n is incomplete, it follows from the Incomplete Set Condition 4.6, that there is a formula $B \in \mathcal{F}$ such that $\Delta_n \nvdash B$ and the set $\Delta_n \cup \{B\}$ is consistent.

Let

$$B_n$$

be the first formula with this property in the sequence 48 of all formulas.

We then define

$$\Delta_{n+1} = \Delta_n \cup \{B_n\}. \tag{51}$$

By the definition, $\Delta \subseteq \Delta_n \subseteq \Delta_{n+1}$ and the set Δ_{n+1} is consistent. Hence Δ_{n+1} is an incomplete consistent extension of Δ.

By the principle of mathematical induction we have defined an infinite sequence

$$\Delta = \Delta_1 \subseteq \Delta_2 \subseteq ..., \subseteq \Delta_n \subseteq \Delta_{n+1} \subseteq .... \tag{52}$$

such that for all $n \in N$, Δ_n is consistent, and moreover, it is an incomplete consistent extension of Δ.

Moreover, we have also defined a sequence

$$B_1, B_2, ..., B_n, ... \tag{53}$$

of formulas, such that for all $n \in N$, $\Delta_n \nvdash B_n$, and the set $\Delta_n \cup \{B_n\}$ is consistent.
Observe that $B_n \in \Delta_{n+1}$ for all $n \geq 1$.

Definition of Δ^*

Now we are ready to define Δ^*, i.e. we define:

$$\Delta^* = \bigcup_{n \in N} \Delta_n. \tag{54}$$

To complete the proof our theorem we have now to prove that Δ^* is a complete consistent extension of Δ. Obviously, by the definition, Δ^* is an extension of Δ. Now we prove (by contradiction) that

FACT 1

Δ^* is consistent.

Proof: assume that Δ^* is inconsistent. By the Finite Inconsistency theorem 4.1 there is a finite subset Δ_0 of Δ^* that is inconsistent. By definition 54 have that

$$\Delta_0 = \{C_1, ..., C_n\} \subseteq \bigcup_{n \in N} \Delta_n.$$

By the definition, $C_i \in \Delta_{k_i}$ for certain Δ_{k_i} in the sequence 52 and $1 \leq i \geq n$. Hence $\Delta_0 \subseteq \Delta_m$ for $m = max\{k_1, k_2, ..k_n\}$. But all sets of the sequence 52 are consistent. This contradicts the fact that Δ_m is inconsistent, as it contains an inconsistent subset Δ_0. Hence Δ^* must be consistent.

FACT 2

Δ^* is complete.

Proof: assume that Δ^* is not complete. By the Incomplete Set Condition 4.6, there is a formula $B \in \mathcal{F}$ such that $\Delta^* \nvdash B$ and the set $\Delta^* \cup \{B\}$ is consistent.

But, by 'refdset, the above condition means that for every $n \in N$, $\Delta_n \nvdash B$ holds and the set $\Delta_n \cup \{B\}$ is consistent.

Since the formula B is one of the formulas of the sequence 48 and it would have to be one of the formulas of the sequence 53, i.e. $B = B_j$ for certain j. Since $B_j \in \Delta_{j+1}$, it proves that $B \in \Delta^* = \bigcup_{n\in N}$. But this means that $\Delta^* \vdash B$, contrary to the assumption.

This proves that Δ^* is a complete consistent extension of Δ and completes the proof out our lemma.

Now we are ready to prove the completeness theorem for the system S.

4.0.4 Proof of the Completeness Theorem

As by assumption our system S is sound, we have to prove only the Completeness part of the Completeness Theorem 2.1, i.e

$$\text{If} \quad \models A, \quad \text{then} \quad \vdash A$$

for any formula A.

We prove it by proving the opposite implication

$$\text{If} \quad \nvdash A, \quad \text{then} \quad \not\models A.$$

We remind that $\not\models A$ means that there is a variable assignment $v : VAR \longrightarrow \{T, F\}$, such that $v^*(A) \neq T$. In classical case it means that $v^*(A) = F$, i.e. that there is a variable assignment that falsifies A . Such v is also called a **counter-model** for A.

Assume that A doesn't have a proof in S, we want to define a **counter-model** for A.

But if $\nvdash A$, then by the lemma 4.4, the set $\{\neg A\}$ is consistent. By the Main Lemma 4.7 there is a complete, consistent extension of the set $\{\neg A\}$, i.e. there is a set set Δ^* such that $\{\neg A\} \subseteq \Delta^*$, i.e.

$$\neg A \in \Delta^*. \tag{55}$$

Since Δ^* is a consistent, complete set, it satisfies the following form of the consistency condition 45, which says that for any A, $\Delta^* \nvdash A$ or $\Delta^* \nvdash \neg A$. It also satisfies the completeness condition 46, which says that for any A, $\Delta^* \vdash A$ or $\Delta^* \vdash \neg A$. This means that for any A, exactly one of the following conditions is satisfied: $\Delta^* \vdash A$, $\Delta^* \vdash \neg A$. In particular, for every propositional variable $a \in VAR$ exactly one of the following conditions is satisfied: $\Delta^* \vdash a$, $\Delta^* \vdash \neg a$. This justifies the correctness of the following definition.

Definition of v

We define the variable assignment

$$v : VAR \longrightarrow \{T, F\} \tag{56}$$

as follows:

$$v(a) = \begin{cases} T & \text{if } \Delta^* \vdash a \\ F & \text{if } \Delta^* \vdash \neg a. \end{cases}$$

We show, as a separate lemma below, that such defined variable assignment v has the following property.

Lemma 4.8 (Property of v)

Let v be the variable assignment defined by (56) and v^* its extension to the set $\mathcal{F}$ of all formulas. Then for every formula $B \in \mathcal{F}$,

$$v^*(B) = \begin{cases} T & \textit{if } \Delta^* \vdash B \\ F & \textit{if } \Delta^* \vdash \neg B. \end{cases} \tag{57}$$

Given the above property (57) of v (still to be proven), we prove that the v is in fact, a counter model for any formula A, such that $\nvdash A$ as follows. Let A be such that $\nvdash A$. By (55), $\neg A \in \Delta^*$ and obviously, $\Delta^* \vdash \neg A$. Hence, by the property (57) of v, $v^*(A) = F$, what proves that v is a counter-model for A and hence ends the proof of the completeness theorem.

In order to really complete the proof we still have to show the lemma 4.8.

Proof of the lemma

The proof is conducted by the induction on the degree of the formula A.

If A is a propositional variable, then the lemma is true holds by (56), i.e. by the definition of v.

If A is not a propositional variable, then A is of the form $\neg C$ or $(C \Rightarrow D)$, for certain formulas C, D. By the inductive assumption the lemma, i.e. the property (57) holds for the formulas C and D.

Case $A = \neg C$. We have to consider two possibilities: $\Delta^* \vdash A$ and $\Delta^* \vdash \neg A$.

Assume $\Delta^* \vdash A$. It means that $\Delta^* \vdash \neg C$. Then from the fact that Δ^* is *consistent* it must be that $\Delta^* \nvdash C$. This means, by the inductive assumption,

that $v^*(C) = F$, and accordingly

$$v^*(A) = v^*(\neg C) = \neg v^*(C) = \neg F = T.$$

Assume now that $\Delta^* \vdash \neg A$. Then from the fact that Δ^* is *consistent* it must be that $\Delta^* \nvdash A$. I.e. $\Delta^* \nvdash \neg C$. If so, then $\Delta^* \vdash C$, as the set Δ^* is *complete.* Hence by the inductive assumption, that $v^*(C) = T$, and accordingly

$$v^*(A) = v^*(\neg C) = \neg v^*(C) = \neg T = F.$$

Thus A satisfies the property 57.

Case $A = (C \Rightarrow D)$**.** As in the previous case, we assume that the lemma, i.e. the property 57 holds for the formulas C, D and we consider two possibilities: $\Delta^* \vdash A$ and $\Delta^* \vdash \neg A$.

Assume $\Delta^* \vdash A$. It means that $\Delta^* \vdash (C \Rightarrow D)$. If at the same time $\Delta^* \nvdash C$, then $v^*(C) = F$, and accordingly

$$v^*(A) = v^*(C \Rightarrow D) = v^*(C) \Rightarrow v^*(D) = F \Rightarrow v^*(D) = T.$$

If at the same time $\Delta^* \vdash C$, then, since $\Delta^* \vdash (C \Rightarrow D)$, we infer, by Modus Ponens, that $\Delta^* \vdash D$. If so, then

$$v^*(C) = v^*(D) = T,$$

and accordingly

$$v^*(A) = v^*(C \Rightarrow D) = v^*(C) \Rightarrow v^*(D) = T \Rightarrow T = T.$$

Thus, if $\Delta^* \vdash A$, then $v^*(A) = T$.

Assume now, as before, that $\Delta^* \vdash \neg A$. Then from the fact that Δ^* is *consistent* it must be that $\Delta^* \nvdash A$, i.e.,

$$\Delta^* \nvdash (C \Rightarrow D).$$

It follows from this that

$$\Delta^* \nvdash D,$$

for if $\Delta^* \vdash D$, then, as $(D \Rightarrow (C \Rightarrow D))$ is assumed to be provable formula 2 in S, by monotonicity

$$\Delta^* \vdash (D \Rightarrow (C \Rightarrow D)).$$

Applying Modus Ponens we obtain $\Delta^* \vdash (C \Rightarrow D)$, which is contrary to the assumption.

Also we must have

$$\Delta^* \vdash C,$$

for otherwise, by the fact that Δ^* we would have

$$\Delta^* \vdash \neg C.$$

But this is impossible, since the formula $(\neg C \Rightarrow (C \Rightarrow D))$ is is assumed to be provable formula 10 in S and by monotonicity

$$\Delta^* \vdash (\neg C \Rightarrow (C \Rightarrow D)).$$

Applying Modus Ponens we would get $\Delta^* \vdash (C \Rightarrow D)$, which is contrary to the assumption. This ends the proof of the lemma and completes the counter-model existence proof of the Completeness Theorem.