The following is a sample of problems from previous exams.

1. **Predicate Logic Models**

Consider the following sentences.

\[ \phi_1: \forall x \neg P(x, x) \]
\[ \phi_2: \forall x \forall y \forall z \left[ P(x, y) \land P(y, z) \rightarrow P(x, z) \right] \]
\[ \phi_3: \forall x \forall y \left[ P(x, y) \rightarrow \exists z \left( P(x, z) \land P(z, y) \right) \right] \]
\[ \phi_4: \neg \forall x \exists y P(x, y) \]

(a) Give a model \( M_1 \) such that \( M_1 \models \phi_1 \land \phi_2 \land \phi_3 \land \phi_4 \).

(b) Give a model \( M_2 \) such that \( M_2 \models \phi_1 \land \phi_2 \land \phi_3 \land \neg \phi_4 \).

2. **Natural deduction**

Give natural deduction proofs for the following sequents.

(a) \( P \rightarrow Q \vdash (P \rightarrow \neg Q) \rightarrow \neg P \)

(b) \( (P \lor Q) \lor (R \rightarrow S) \vdash (S \lor Q) \lor (R \rightarrow P) \)
   
   Hint. Use the \( \lor \)-rule and LEM.

3. **Natural deduction**

Use the standard natural deduction rules to prove the following sequents.

(a) \( \forall y P(c, y), \forall x \forall y [P(x, y) \rightarrow P(f(x), f(y))] \vdash \exists z [P(c, z) \land P(z, f(f(c)))] \)

   (where \( P \) is a binary predicate symbol, \( f \) a unary function symbol, and \( c \) a constant).

(b) \( \exists x P(x, x), \forall x [\exists y P(x, y) \rightarrow Q(x)] \vdash \exists x Q(x) \)

4. **Unification**

(a) Determine whether the unification problem

\[ \{ x =^? f(y, g(y)), g(f(z, a)) =^? g(y) \} \]

is solvable. Give a most general unifier or else explain why there is no unifier. (Note that \( a \) is a constant, whereas \( x \), \( y \), and \( z \) are variables.)
(b) Give terms \( s, t, \) and \( u \) such that (i) \( s \) and \( t \) have exactly one unifier, (ii) \( t \) and \( u \) have infinitely many unifiers, and (iii) \( s \) and \( u \) are not unifiable.

\[
\begin{align*}
s &= \\
t &= \\
u &=
\end{align*}
\]

5. Clause Logic

Recall that a clause is a disjunction of literals (atomic formulas or negations thereof). Find unsatisfiable sets of clauses as specified. If there is no unsatisfiable set of clauses that meets the stated restrictions, explain why.

(a) Each clause in \( S_1 \) must contain both positive and negative literals. For example, \( p(x) \lor \neg q(y) \) qualifies but \( p(x) \) and \( \neg p(x) \lor \neg q(y) \) do not.

(b) No clause in \( S_2 \) must contain both positive and negative literals. For example, \( p(x) \lor q(y) \) qualifies but \( \neg p(x) \lor q(y) \) does not.

6. Skolemization

(a) Let \( \phi \) be the sentence \( \neg [\exists x P(x, x) \lor \neg \exists y \exists z \forall x (P(x, y) \lor P(x, z))] \).

i. Convert \( \phi \) to prenex form.

ii. Skolemize the formula you obtained in the first part.

iii. Use resolution to determine whether the skolemized formula from the second part is unsatisfiable.

7. Resolution

Use resolution to determine whether the following set of clauses is unsatisfiable:

\[
\begin{align*}
\neg P(x) \lor Q(x) \lor R(x, f(x)), & \quad (1) \\
\neg P(x) \lor Q(x) \lor S(f(x)) & \quad (2) \\
\neg R(a, x) \lor T(x), & \quad (3) \\
\neg T(x) \lor \neg Q(x), & \quad (4) \\
\neg T(x) \lor \neg S(x), & \quad (5) \\
P(a), & \quad (6) \\
T(a). & \quad (7)
\end{align*}
\]
8. Compactness

(a) Given a sentence $\phi_k$ that expresses that “there are at least $k$ distinct elements.”

(b) Prove that there is no set $S$ of predicate logic formulas that is satisfied exactly by those models that have a finite domain.