Prenex form.

A possible prenex form of

$$\neg \exists x ((\forall y \forall z P(y, z)) \land \neg P(x, z))$$

is

$$\forall x \exists y \exists z (\neg P(y, z) \lor P(x, z)).$$

Logical equivalence.

The two sentences $$\forall x \exists y (P(x) \land Q(y))$$ and $$\exists y \forall x (P(x) \land Q(y))$$ are equivalent, as the following proof shows:

<table>
<thead>
<tr>
<th>Step</th>
<th>Formula</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$$\forall x \exists y (P(x) \land Q(y))$$</td>
</tr>
<tr>
<td>2</td>
<td>$$\sim \forall x (P(x) \land \exists y Q(y))$$</td>
</tr>
<tr>
<td>3</td>
<td>$$\sim \forall x P(x) \land \exists y Q(y)$$</td>
</tr>
<tr>
<td>4</td>
<td>$$\sim \exists y (\forall x P(x) \land Q(y))$$</td>
</tr>
<tr>
<td>5</td>
<td>$$\sim \exists y \forall x (P(x) \land Q(y))$$</td>
</tr>
</tbody>
</table>

Logical equivalence.

The two sentences $$\forall x \exists y P(x, y)$$ and $$\exists y \forall x P(y, x)$$ are not equivalent. Consider a model $$\mathcal{M}$$ with the set of (negative and nonnegative) integers as universe, where $$P^\mathcal{M}$$ is the less-than relation. The first sentence (which asserts that every integer is less than some other integer) is true in this model, but the second sentence (which states that there is a smallest integer) is false.

Logical equivalence.

Consider $$\forall x \exists x P(x, x)$$ and $$\exists x \forall x P(x, x)$$. Since $$\forall x \exists x P(x, x)$$ is logically equivalent to $$\exists x P(x, x)$$, whereas $$\exists x \forall x P(x, x)$$ is equivalent to $$\forall x P(x, x)$$, the two formulas are not equivalent.

Logical consequence.

The sentence $$\exists x (P(x) \land R(x))$$ is not a logical consequence of $$\exists x (P(x) \land Q(x))$$ and $$\exists x (Q(x) \land R(x))$$. 
For instance, consider a model $M$ with domain $\{a, b\}$, where $P^M = \{a\}$, $Q^M = \{a, b\}$, and $R^M = \{b\}$. Then $\exists x (P(x) \land Q(x))$ is true in $M$ as $P(x)$ and $Q(x)$ both evaluate to true if $a$ is assigned to $x$. Similarly, $\exists x (Q(x) \land R(x))$ is true in $M$ as $Q$ and $R$ both evaluate to true if $b$ is assigned to $x$. But $\exists x (P(x) \land R(x))$ is not true in $M$, as there is no assignment to $x$ for which both $P(x)$ and $R(x)$ evaluate to true at the same time.

**Skolemization.**

We skolemize various sentences.

1. $\exists x \forall y \exists z (P(x, y) \land P(y, z) \rightarrow P(x, z))$
   
   **Solution:** $\forall y (P(x, y) \land P(y, f(y)) \rightarrow P(x, f(y)))$

2. $\forall x \forall y (P(x, y) \rightarrow \exists z (P(x, z) \rightarrow P(z, y)))$
   
   **Solution:** $\forall x \forall y (P(x, y) \rightarrow (P(x, f(x, y)) \rightarrow P(f(x, y), y)))$

3. $\forall x \exists z P(x, x)$
   
   **Solution:** $\forall x P(f(x), f(x))$

4. $\exists x \forall x P(x, x)$
   
   **Solution:** $\forall x P(x, x)$

**Prenex form and Skolemization.**

We convert the following formula to a set of clauses so that satisfiability is preserved:

$$\neg(\forall x \exists y P(x, y) \rightarrow (\forall y \exists z \neg Q(x, z) \land \forall y \forall z R(y, z))).$$

First we rename bound variables so that different quantifiers bind different variables and no variable has both free and bound occurrences:

$$\neg(\forall u \exists v P(u, v) \rightarrow (\forall y \exists z \neg Q(x, z) \land \forall s \forall t R(s, t))).$$

Next observe that this formula is satisfiable if, and only if, its existential closure is satisfiable:

$$\exists x [ \neg(\forall u \exists v P(u, v) \rightarrow (\forall y \exists z \neg Q(x, z) \land \forall s \forall t R(s, t))]].$$

Conversion to prenex form takes several steps; one intermediate formula is

$$\exists x [\forall u \exists v P(u, v) \land (\exists y \forall z Q(x, z) \lor \exists s \forall t R(s, t))].$$
A possible prenex formula is
\[ \exists x \forall u \exists v \forall z \exists s \forall t \, (P(u, v) \land (Q(x, z) \lor R(s, t))). \]

Skolemization yields a universal formula,
\[ \forall u \forall z \forall t \, (P(u, f_r(u)) \land (Q(c_x, z) \lor R(f_s(u, z), t))), \]
where \( c_x \) and \( f_s \) are Skolem symbols. (Other universal sentences can also be obtained from the given initial formula.) The corresponding clauses are \( P(u, f_r(u)) \) and \( Q(c_x, z) \lor R(f_s(u, z), t)) \).

**Substitution.**

Let \( \sigma_1 \) be the substitution \([x \mapsto y, y \mapsto z, z \mapsto x]\), \( \sigma_2 \) the substitution \([x \mapsto y, y \mapsto z, z \mapsto y]\), and \( \sigma_3 \) the substitution \([x \mapsto x + y, y \mapsto y + z, z \mapsto y]\).

Since \( \sigma_2 = \sigma_1[x \mapsto y] \), the substitution \( \sigma_1 \) is more general than \( \sigma_2 \). We also have \( \sigma_3 = \sigma_1[x \mapsto x + z, y \mapsto x + y, z \mapsto y + z] \) so that \( \sigma_1 \) is more general than \( \sigma_3 \). (But neither \( \sigma_2 \) nor \( \sigma_3 \) is more general than \( \sigma_1 \).)

We also have
\[
\begin{align*}
\sigma_1 \sigma_2 &= [x \mapsto z, z \mapsto y] \\
\sigma_2 \sigma_2 &= [x \mapsto z] \\
\sigma_2 \sigma_3 &= [x \mapsto y + z, y \mapsto x + z, z \mapsto y + z] \\
\sigma_1 \sigma_2 \sigma_3 &= [x \mapsto x + z, y \mapsto y + z, z \mapsto y + z]
\end{align*}
\]

**Unification.**

The unification problem \( \{ x =^? f(y, g(y)), g(f(z, a)) =^? g(y) \} \) is solvable. The derivation,
\[
\begin{align*}
x &=^? f(y, g(y)), g(f(z, a)) =^? g(y) \\
\Rightarrow \text{DECOMPOSE} & \quad x =^? f(y, g(y)), f(z, a) =^? y \\
\Rightarrow \text{ORIENT} & \quad x =^? f(y, g(y)), y =^? f(z, a) \\
\Rightarrow \text{ELIMINATE} & \quad x =^? f(f(z, a), g(f(z, a))), y =^? f(z, a)
\end{align*}
\]
yields a most general unifier, \([x \mapsto f(f(z, a), g(f(z, a))), y \mapsto f(z, a)]\).

Another unifier, but not a most general one, is \([x \mapsto f(f(a, a), g(f(a, a))), y \mapsto f(a, a)]\).
Unification.

The unification problem

\[ f(x, g(a, y)) = ? f(h(y), g(y, a)), g(x, h(y)) = ? g(z, z) \]

where \( x, y, \) and \( z \) are the only variables (and all other symbols denote functions or constants), is solvable. The derivation,

\[
\begin{align*}
  f(x, g(a, y)) &= f(h(y), g(y, a)), g(x, h(y)) &= g(z, z) \\
  \Rightarrow \text{DECOMPOSE} & \quad x = ? h(y), g(a, y) = ? g(y, a), g(x, h(y)) = ? g(z, z) \\
  \Rightarrow \text{DECOMPOSE} & \quad x = ? h(y), a = ? y, y = ? a, g(x, h(y)) = ? g(z, z) \\
  \Rightarrow \text{ELIMINATE} & \quad x = ? h(a), a = ? y, y = ? a, g(x, h(a)) = ? g(z, z) \\
  \Rightarrow \text{DELETE} & \quad \text{Eliminate variables} \\
  \Rightarrow \text{ELIMINATE} & \quad x = ? h(a), y = ? a, g(x, h(a)) = ? g(z, z) \\
  \Rightarrow \text{DECOMPOSE} & \quad x = ? h(a), y = ? a, h(a) = ? z \\
  \Rightarrow \text{ORIENT} & \quad x = ? h(a), y = ? a, z = ? h(a)
\]

yields a most general unifier,

\[[x \mapsto h(a), y \mapsto a, z \mapsto a].\]

Unification.

The unification problem \( \{ x_1 = ? f(x_2), x_2 = ? f(x_3), g(x_4) = ? x_3, g(x_1) = ? x_4 \} \) is not solvable: after applying several orientation and elimination steps to the given set, one obtains a unification problem to which the occurs-check rule applies.

Ground resolution.

We use ground resolution to show that the set of clauses

\[ \{ P \lor \neg Q, P \lor R, \neg Q \lor R, \neg P \lor Q, Q \lor \neg R, \neg P \lor \neg R \} \]

is unsatisfiable. Here is one possible derivation of a contradiction:

\[
\begin{align*}
P \lor \neg Q & \quad \text{given} \quad (1) \\
P \lor R & \quad \text{given} \quad (2) \\
\neg Q \lor R & \quad \text{given} \quad (3) \\

\neg P \lor Q & \quad \text{given} \quad (4)
\]

\[ P \lor \neg Q \]

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\[ P \lor \neg Q \]
\[
Q \lor \neg R \quad \text{given} \quad (5)
\]
\[
\neg P \lor \neg R \quad \text{given} \quad (6)
\]
\[
P \lor Q \quad \text{RES 2,5} \quad (7)
\]
\[
\neg P \lor \neg Q \quad \text{RES 3,6} \quad (8)
\]
\[
P \lor P \quad \text{RES 1,7} \quad (9)
\]
\[
P \quad \text{FACT 9} \quad (10)
\]
\[
\neg P \lor \neg P \quad \text{RES 4,8} \quad (11)
\]
\[
\neg P \quad \text{FACT 11} \quad (12)
\]
\[
\bot \quad \text{RES 10,12} \quad (13)
\]

**Ground resolution.**

Let \( N \) be the set containing the following (ground) clauses:

\[
\neg P \lor Q \lor R \quad (14)
\]
\[
P \lor \neg R \quad (15)
\]
\[
Q \lor \neg R \quad (16)
\]
\[
P \lor R \lor \neg S \quad (17)
\]
\[
\neg P \lor T \quad (18)
\]
\[
\neg Q \lor R \lor T \quad (19)
\]
\[
Q \lor R \lor S \lor T \quad (20)
\]
\[
\neg Q \lor \neg T \quad (21)
\]
\[
P \lor S \lor \neg T \quad (22)
\]

We derive new clauses by resolution:

\[
P \lor Q \lor R \lor S \lor S \quad [7 \text{ and } 9] \quad (23)
\]
\[
P \lor \neg Q \lor R \lor S \quad [6 \text{ and } 9] \quad (24)
\]
\[
\neg Q \lor \neg Q \lor R \quad [6 \text{ and } 8] \quad (25)
\]
\[
\neg P \lor \neg Q \quad [5 \text{ and } 8] \quad (26)
\]
\[
P \lor P \lor Q \lor R \lor R \quad [10 \text{ and } 4] \quad (27)
\]
\[
P \lor \neg Q \lor \neg Q \quad [12 \text{ and } 2] \quad (28)
\]
\[
\neg P \lor Q \lor Q \quad [1 \text{ and } 3] \quad (29)
\]
\[
P \lor P \lor Q \lor Q \lor Q \quad [14 \text{ and } 3] \quad (30)
\]
\[
\neg P \lor \neg P \quad [16 \text{ and } 13] \quad (31)
\]
\[
P \lor P \lor P \quad [17 \text{ and } 15] \quad (32)
\]
\[
\bot \quad [18 \text{ and } 19] \quad (33)
\]
Since a contradiction has been derived the initial set $N$ is unsatisfiable.

**Ground resolution.**

We derive a contradiction from the following clauses using resolution:

<table>
<thead>
<tr>
<th>Clause</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>$P_{1,1} \lor P_{1,2}$</td>
<td></td>
</tr>
<tr>
<td>$P_{1,1} \lor \neg P_{2,1}$</td>
<td></td>
</tr>
<tr>
<td>$P_{1,2} \lor \neg P_{2,2}$</td>
<td></td>
</tr>
<tr>
<td>$P_{2,1} \lor P_{2,2}$</td>
<td></td>
</tr>
<tr>
<td>$P_{3,1} \lor P_{3,2}$</td>
<td></td>
</tr>
<tr>
<td>$\neg P_{1,1} \lor \neg P_{2,1}$</td>
<td></td>
</tr>
<tr>
<td>$\neg P_{1,1} \lor \neg P_{3,1}$</td>
<td></td>
</tr>
<tr>
<td>$\neg P_{1,2} \lor \neg P_{3,2}$</td>
<td></td>
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<tr>
<td>$\neg P_{2,1} \lor \neg P_{3,1}$</td>
<td></td>
</tr>
<tr>
<td>$\neg P_{2,2} \lor \neg P_{3,2}$</td>
<td></td>
</tr>
</tbody>
</table>

In each inference the maximal literals in each premise were resolved, where maximality is determined by the following ordering: using the following order on literals:

$\neg P_{3,2} \succ P_{3,2} \succ \neg P_{3,1} \succ P_{3,1} \succ \neg P_{2,2} \succ \cdots \succ \neg P_{1,1} \succ P_{1,1}$.

(This is also known as “ordered resolution.”) Factoring has been systematically applied to eliminate multiple occurrences of the same literal from a clause, and for simplicity only clauses without multiple occurrences of the same literal are listed. The first nine clauses are given.

<table>
<thead>
<tr>
<th>Clause</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>$P_{1,1} \lor P_{1,2}$</td>
<td>[given]</td>
</tr>
<tr>
<td>$P_{2,1} \lor P_{2,2}$</td>
<td>[given]</td>
</tr>
<tr>
<td>$P_{3,1} \lor P_{3,2}$</td>
<td>[given]</td>
</tr>
<tr>
<td>$\neg P_{1,1} \lor \neg P_{2,1}$</td>
<td>[given]</td>
</tr>
<tr>
<td>$\neg P_{1,2} \lor \neg P_{2,2}$</td>
<td>[given]</td>
</tr>
<tr>
<td>$\neg P_{1,1} \lor \neg P_{3,1}$</td>
<td>[given]</td>
</tr>
<tr>
<td>$\neg P_{1,2} \lor \neg P_{3,2}$</td>
<td>[given]</td>
</tr>
<tr>
<td>$\neg P_{2,1} \lor \neg P_{3,1}$</td>
<td>[given]</td>
</tr>
<tr>
<td>$\neg P_{2,2} \lor \neg P_{3,2}$</td>
<td>[given]</td>
</tr>
<tr>
<td>$\neg P_{2,2} \lor P_{3,1}$</td>
<td>3 &amp; 9</td>
</tr>
<tr>
<td>$P_{1,2} \lor P_{3,1}$</td>
<td>3 &amp; 7</td>
</tr>
<tr>
<td>$\neg P_{1,2} \lor \neg P_{2,2}$</td>
<td>10 &amp; 8</td>
</tr>
<tr>
<td>$\neg P_{1,1} \lor \neg P_{2,2}$</td>
<td>10 &amp; 6</td>
</tr>
<tr>
<td>$\neg P_{1,2} \lor \neg P_{2,1}$</td>
<td>11 &amp; 8</td>
</tr>
<tr>
<td>$\neg P_{1,1} \lor \neg P_{1,2}$</td>
<td>11 &amp; 6</td>
</tr>
<tr>
<td>$\neg P_{1,2} \lor P_{2,1}$</td>
<td>2 &amp; 5</td>
</tr>
<tr>
<td>$\neg P_{1,1} \lor P_{2,1}$</td>
<td>2 &amp; 13</td>
</tr>
</tbody>
</table>
\(-P_{1,2}\) 16 & 14, plus factoring \((18)\)
\(-P_{1,1} \lor -P_{1,2}\) 16 & 4 \((19)\)
\(-P_{1,1}\) 17 & 4, plus factoring \((20)\)
\(P_{1,1}\) 1 & 18 \((21)\)
\(\bot\) 21 & 20 \((22)\)

**Instantiation of clauses.**

Consider the following clauses,

\[-R(x, x)\] \((1)\)
\[-R(x, y) \lor R(f(x), y)\] \((2)\)
\(R(x, f(x))\) \((3)\)

Suitable instantiation yields a set of ground clauses,

\[-R(f(a), f(a))\] \((1')\)
\[-R(a, f(a)) \lor R(f(a), f(a))\] \((2')\)
\(R(a, f(a))\) \((3')\)

that is unsatisfiable, as one can obtain a contradiction by two steps of resolution. Hence, the initial set of clauses is also unsatisfiable.

**Resolution.**

Consider the following clauses:

\[-R(x, y) \lor -R(y, x)\]
\(R(fx, fx)\)

We apply resolution to the first clause and a renamed version (renaming \(x\) to \(x'\)) of the second clause, using most general unifier \(\sigma = [x \mapsto fx', y \mapsto fx']\), to obtain

\[-R(fx', fx')\).

From the (original) second clause and the new clause we obtain a contradiction by applying resolution with most general unifier \(\sigma = [x \mapsto x']\). The initial set of clauses is therefore not satisfiable.

**Resolution.**
We use resolution to show that the set of two clauses,

\[ \neg R(x, y) \lor \neg R(y, x) \\
R(ffx, fy) \]

is unsatisfiable. After renaming \( x \) to \( x' \) and \( y \) to \( y' \) in the second clause, we apply resolution to the two given clauses to obtain

\[ \neg R(fy', ffx') \]

by using the unifier \( \sigma = [x \mapsto ffx', y \mapsto fy'] \). From the second clause and this new clause we get a contradiction by applying resolution with unifier \( \sigma = [y \mapsto fx', y' \mapsto fx] \). The initial set of clauses is therefore not satisfiable.

Resolution.

Consider the set of three clauses,

\[ \neg R(x, y) \lor \neg R(y, z) \lor R(x, z) \\
\neg R(fx, ffx) \\
R(x, fx) \]

We rename \( x \) to \( x' \) in the second clause and apply resolution with most general unifier \( \sigma = [x \mapsto fx', z \mapsto ffx'] \) to the renamed clause and the first clause, to obtain

\[ \neg R(fx', y) \lor \neg R(y, ffx'). \]

Applying resolution to the third and fourth clause we get

\[ \neg R(ffx', ffx') \]

using the most general unifier \( [x \mapsto fx', y \mapsto ffx'] \).

From the third and fifth clause we obtain a contradiction by resolution via most general unifier \( [x \mapsto ffx'] \). The initial set of clauses is therefore not satisfiable.

Resolution.

We use resolution and factoring to show that the following set of clauses is unsatisfiable:

\[ \neg P(x, y) \lor \neg P(y, x) \lor \neg P(x, a) \\
P(x, a) \lor P(x, f(x)) \\
P(x, a) \lor P(f(x), x) \]
where \(a\) is a constant and \(x\) and \(y\) are variables. Here is one possible derivation of a contradiction:

\[
\begin{align*}
-P(x, y) \lor -P(y, x) \lor -P(x, a) & \quad \text{given} \quad (1) \\
P(x, a) \lor P(x, f(x)) & \quad \text{given} \quad (2) \\
P(x, a) \lor P(f(x), x) & \quad \text{given} \quad (3) \\
-P(x, x) \lor -P(x, a) & \quad \text{FACT 1} \quad [\ y \mapsto x \] \quad (4) \\
-P(a, a) & \quad \text{FACT 4} \quad [x \mapsto a] \quad (5) \\
P(a, f(a)) & \quad \text{RES 2,5} \quad [x \mapsto a] \quad (6) \\
P(f(a), a) & \quad \text{RES 3,5} \quad [x \mapsto a] \quad (7) \\
-P(f(a), f(a)) \lor -P(y, f(a)) & \quad \text{RES 1,7} \quad [x \mapsto f(a)] \quad (8) \\
-P(a, f(a)) & \quad \text{RES 7,8} \quad [y \mapsto a] \quad (9) \\
\bot & \quad \text{RES 6,9} \quad (10)
\end{align*}
\]

**Resolution.**

We use resolution to determine whether

\[
\eta : \forall x \exists y \exists z [R(f(x), y) \lor R(y, f(z))]
\]

is a logical consequence of

\[
\phi : \forall x \exists y [R(x, f(y)) \rightarrow R(y, f(x))]
\]

and

\[
\psi : \exists x \forall y \exists z [\neg R(x, f(y)) \rightarrow \neg R(y, f(z))].
\]

First note that \(\eta\) is a logical consequence of \(\phi\) and \(\psi\) if, and only if, the implication \(\phi \land \psi \rightarrow \eta\) is valid. The latter problem is equivalent to determining whether \(\phi \land \psi \land \neg \eta\) is unsatisfiable.

We next skolemize \(\phi\), \(\psi\), and \(\neg \eta\) to obtain universal sentences,

\[
\begin{align*}
\phi' : \forall x [R(x, f(g(x))) \rightarrow R(g(x), f(x))] \\
\psi' : \forall y [\neg R(c, f(y)) \rightarrow \neg R(y, f(h(y)))] \\
\eta' : \forall y [\neg R(f(d), y) \lor R(y, f(i(y)))]
\end{align*}
\]

where \(c, d, g, h, \) and \(i\) denote Skolem functions. The formula \(\phi' \land \psi' \land \neg \eta'\) is unsatisfiable if, and only if, \(\phi' \land \psi' \land \eta'\) is unsatisfiable. The latter
formula is unsatisfiable if, and only if, the following set of clauses $S$ is unsatisfiable:

$$
\neg R(x, f(g(x))) \lor R(g(x), f(x)) \\
R(c, f(y)) \lor \neg R(y, f(h(y))) \\
\neg R(f(d), y) \\
\neg R(y, f(i(y)))
$$

Each clause in $S$ contains a negative literal. In general, if both premises of a resolution inference contain a negative literal, so does the conclusion; and, similarly, if factoring is applied to a clause with a negative literal, the conclusion also contains a negative literal. Thus, we can only derive clauses with negative literals from $S$ (by resolution and factoring), but not the empty clause (a contradiction). We may conclude that $S$ is satisfiable and, hence, $\eta$ is not a logical consequence of $\phi$ and $\psi$. 