LECTURE 3b
Chapter 3
Propositional Semantics: Classical and Many Valued

Extensional Semantics
Extensional Semantics $M$ - Introduction

Given a propositional language $L_{CON}$, the symbols for its connectives always have some intuitive meaning.

A formal definition of the meaning of these symbols is called a semantics for the language $L_{CON}$.

A given language $L_{CON}$ can have different semantics but we always define them in order to single out special formulas of the language, called tautologies.

Tautologies are formulas of the language that are always true under a given semantics.
We have already introduced the intuitive and formal notions of a classical semantics, discussed its motivation and underlying assumptions.

The classical semantics assumption is that it considers only two logical values. The other one is that all classical propositional connectives are extensional.

We have also observed that in everyday language there are expressions such as "I believe that", "it is possible that", "certainly", etc.... and that they are represented by some propositional connectives which are not extensional.
Non-extensional connectives do not play any role in mathematics and so are not discussed in classical logic and will be studied separately.

The extensional connectives are defined intuitively as such that the logical value of the formulas form by means of these connectives and certain given formulas depends only on the logical value(s) of the given formulas.
We adopt a following formal definition of extensional connectives for a propositional language $\mathcal{L}_{CON}$.

**Definition**

Let $\mathcal{L}_{CON}$ be such that $CON = C_1 \cup C_2$, where $C_1, C_2$ are the sets of unary and binary connectives, respectively.

Let $LV$ be a non-empty set of logical values.

A connective $\nabla \in C_1$ or $\circ \in C_2$ is called extensional if it is defined by a respective function:

$$\nabla : LV \rightarrow LV \quad \text{or} \quad \circ : LV \times LV \rightarrow LV$$
A semantics $\mathcal{M}$ for a language $\mathcal{L}_{\text{CON}}$ is called extensional provided all connectives in $\text{CON}$ are extensional and its notion of tautology is defined terms of the connectives and their logical values.

A semantics with a set of $m$ logical values is called a $m$-valued extensional. The classical semantics is a special case of a 2-valued extensional semantics. Classical semantics defines classical logic with its set of classical propositional tautologies.

Many of logics are defined by various extensional semantics with sets of logical values $\text{LV}$ with more than 2 elements.
The languages of many important logics like modal, multi-modal, knowledge, believe, temporal, contain connectives that are not extensional because they are defined by non-extensional semantics.

The intuitionistic logic is based on the same language as the classical one and its Kripke Models semantics is not extensional.

Defining a semantics for a given language means more then defining connectives.

The ultimate goal of any semantics is to define the notion of its own tautology.
Extensional Semantics $\mathbf{M}$ Introduction

In order to **define** which formulas of a given $L_{CON}$ we want to to be **tautologies** under a given semantics $\mathbf{M}$ we assume that the set $LV$ of logical values of $\mathbf{M}$ always has a **distinguished** logical value, often denoted by $T$ for "absolute" truth.

We also can **distinguish**, and often we do, another special value $F$ representing "absolute" falsehood.

We will use these symbols $T, F$ for "absolute" truth and falsehood.

We may also use other symbols like $1, 0$ or others.
Extensional Semantics M Introduction

The "absolute" truth value $T$ serves to define a notion of a tautology (as a formula always "true")

Extensional semantics share not only the similar pattern of defining their (extensional) connectives, but also the method of defining the notion of a tautology

We hence define a general notion of an extensional semantics as sequence of steps leading to the definition of a tautology
Extensional Semantics \textbf{M} Introduction

Here are the \textbf{steps} leading to the definition of a \textbf{tautology}

**Step 1**  We \textbf{define} all extensional \textbf{connectives} of \textbf{M}

**Step 2**  We \textbf{define} main component of the definition of a \textbf{tautology}, namely a \textbf{function}  \textbf{v}  that assigns to any formula \( A \in F \) its logical \textbf{value} from \textbf{LV}

The function \textbf{v}  is often called a \textbf{truth assignment} and we will use this name
Extensional Semantics $M$ Introduction

**Step 3** Given a truth assignment $v$ and a formula $A \in \mathcal{F}$, we define what does it mean that

$v$ satisfies $A$

i.e. we define a notion saying that $v$ is a model for $A$ under semantics $M$

**Step 4** We define a notion of tautology as follows

$A$ is a tautology under semantics $M$ if and only if all truth assignments $v$ satisfy $A$

i.e. that all truth assignments $v$ are models for $A$
We use a notion of a **model** because it is an important, if not the most important notion of modern **logic**. The notion of a **model** is usually **defined** in terms of the notion of **satisfaction**.

In **classical** propositional logic these notions are the **same** and the **use** of expressions “*v satisfies A*” and “*v is a model for A*” is **interchangeable**.

This also is **true** for of any propositional **extensional semantics** and in particular it holds for **m-valued** semantics discussed later in this chapter.
The notions of **satisfaction** and **model** are not interchangeable for **predicate** languages semantics.

We already discussed **intuitively** the notion of **model** and **satisfaction** for **predicate** language in chapter 2.

We will define them in **full formality** in chapter 8.

The use of the notion of a **model** also allows us to adopt and discuss the **standard** predicate logic **definitions** of **consistency** and **independence** for **propositional** case.
Definition

Any formal definition of an extensional semantics $M$ for a given language $L_{CON}$ consists of specifying the following steps defining its main components:

**Step 1** We define a set $LV$ of logical values, its distinguished value $T$, and define all connectives of $L_{CON}$ to be extensional.

**Step 2** We define notion of a truth assignment and its extension.

**Step 3** We define notions of satisfaction, model, counter model.

**Step 4** We define notion of a tautology under the semantics $M$. 

Extensional Semantics $M$ Formal Definition
Extensional Semantics $\mathcal{M}$ Formal Definition

What differs one semantics from the other is the choice of the set $LV$ of logical values and definition of the connectives of $\mathcal{L}_{CON}$, that are defined in the first step below.

**Step 1** We adopt a following formal definition of extensional connectives of $\mathcal{L}_{CON}$

**Definition**

Let $\mathcal{L}_{CON}$ be such that $CON = C_1 \cup C_2$, where $C_1, C_2$ are the sets of unary and binary connectives, respectively.

Let $LV$ be a non-empty set of logical values.

A connective $\land \in C_1$ or $\circ \in C_2$ is called extensional if it is defined by a respective function

$$\land : LV \rightarrow LV \quad \text{or} \quad \circ : LV \times LV \rightarrow LV$$
Truth Assignment Formal Definition

Step 2 We define a function called truth assignment and its extension in terms of the propositional connectives as defined in the Step 1

Definition

Let $LV$ be the set of logical values of $M$ and $VAR$ the set of propositional variables of the language $L_{CON}$

Any function

$$v : VAR \rightarrow LV$$

is called a truth assignment under semantics $M$

We call it for short a $M$ truth assignment

We use the term $M$ truth assignment and $M$ truth extension to stress that it is defined relatively to a given semantics $M$. 
Definition

Given a $\mathbf{M}$ truth assignment $v : \text{VAR} \rightarrow \text{LV}$ We define its extension $v^*$ to the set $\mathcal{F}$ of all formulas of $\mathcal{L}_{\text{CON}}$ as any function

$$v^* : \mathcal{F} \rightarrow \text{LV}$$

such that the following conditions are satisfied.

(i) for any $a \in \text{VAR}$,

$$v^*(a) = v(a);$$

(ii) For any connectives $\nabla \in C_1$, $\circ \in C_2$, and for any formulas $A, B \in \mathcal{F}$,

$$v^*(\nabla A) = \nabla v^*(A) \quad \text{and} \quad v^*((A \circ B)) = \circ(v^*(A), v^*(B))$$

We call the $v^*$ the $\mathbf{M}$ truth extension.
Remark

The symbols on the left-hand side of the equations

\[ v^*(\nabla A) = \nabla v^*(A) \quad \text{and} \quad v^*(A \circ B) = \circ(v^*(A), v^*(B)) \]

represent connectives in their natural language meaning and the symbols on the right-hand side represent connectives in their semantical meaning as defined in the Step1.
**M Truth Extension Formal Definition**

We use names "**M truth assignment**" and "**M truth extension**" to stress that we define them for the set of logical values of the semantics **M**

**Notation Remark**

For any function \( g \), we use a symbol \( g^* \) to denote its extension to a larger domain.

Mathematician often use the same symbol \( g \) for both a function \( g \) and its extension \( g^* \).
Satisfaction and Model

Step 3  The notions of satisfaction and model are interchangeable in $M$ semantics and we define them as follows.

Definition

Given an $M$ truth assignment $v: \text{VAR} \rightarrow \text{LV}$ and its $M$ truth extension $v^*$ Let $T \in \text{LV}$ be the distinguished logical truth value

We say that the truth assignment $v$ $M$ satisfies a formula $A$ if and only if $v^*(A) = T$

We write symbolically

$$v \models_M A$$

Any truth assignment $v$, such that $v \models_M A$ is called an $M$ model for the formula $A$
Counter Model

Definition

Given an $M$ truth assignment $v : \text{VAR} \rightarrow \text{LV}$ and its $M$ truth extension $v^*$. Let $T \in \text{LV}$ be the distinguished logical truth value.

We say that the truth assignment $v$ $M$ does not satisfy a formula $A$ if and only if $v^*(A) \neq T$.

We write symbolically:

$$v \not\models_M A$$

Any truth assignment $v$, such that $v \not\models_M A$ is called an $M$ counter model for the formula $A$. 


Step 4  We define the notion of **M tautology** as follows

**Definition**

A formula $A$ is an **M tautology** if and only if $v \models_m A$, for all truth assignments $v : VAR \rightarrow LV$. We denote it as $\models_m A$.

We also say that $A$ is an **M tautology** if and only if all truth assignments $v : VAR \rightarrow LV$ are **M models** for $A$. 
Observe that directly from definition of the M model we get the following equivalent form of the definition of tautology

**Definition**

A formula $A$ is an M tautology if and only if $v^*(A) = T$, for all truth assignments $v : \text{VAR} \rightarrow \text{LV}$

We denote by $\text{MT}$ the set of all tautologies under the semantic M, i.e.

$$\text{MT} = \{ A \in \mathcal{F} : \models_M A \}$$
Obviously, when we develop a logic by defining its semantics we want the semantics to be such that the logic has a non empty set of its tautologies. We express it in a form of the following definition:

**Definition**

Given a language $L_{\text{CON}}$ and its extensional semantics $M$, the semantics $M$ is well defined if and only if its set $MT$ of all tautologies is non empty, i.e. when

$$MT \neq \emptyset$$
Extensional Semantics $\mathbb{M}$

As the next steps we use the definitions established here to define and discuss in details the following particular cases of the extensional semantics $\mathbb{M}$

Many valued semantics have their beginning in the work of Łukasiewicz (1920)

He was the first to define a 3-valued extensional semantics for a language $\mathcal{L}_{\{\neg, \wedge, \vee, \Rightarrow\}}$ of classical logic, and called it a 3-valued logic, for short
The other logics defined by various extensional semantics followed and we will discuss some of them. In particular we present Heyting’s 3-valued semantics as an introduction to the discussion of first ever semantics for the intuitionistic logic and some modal logics.
Challenge Exercise

1. Define your own propositional language $L_{\text{CON}}$ that contains also different connectives that the standard connectives $\neg$, $\cup$, $\cap$, $\Rightarrow$

Your language $L_{\text{CON}}$ does not need to include all (if any!) of the standard connectives $\neg$, $\cup$, $\cap$, $\Rightarrow$

2. Describe intuitive meaning of the new connectives of your language

3. Give some motivation for your own semantic $M$

4. Define formally your own extensional semantics $M$ for your language $L_{\text{CON}}$

Write carefully all Steps 1-4 of the definition of your $M$