Introduction to Predicate Logic

Part 2

cse371/ math371/cse541

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Predicate Logic Introduction

Part 2

- Basic Laws of Quantifiers – Predicate Logic Tautologies
- Intuitive Semantics for Predicate Logic

Basic Predicate Logic Tautologies

De Morgan Laws

$$\neg \forall x\, A(x) \equiv \exists x\, \neg A(x)$$

$$\neg \exists x\, A(x) \equiv \forall x\, \neg A(x)$$

where $A(x)$ is any formula with free variable x,

$\equiv$ means "logically equivalent"

Definability of Quantifiers

$$\forall x\, A(x) \equiv \neg \exists x\, \neg A(x)$$

$$\exists x\, A(x) \equiv \neg \forall x\, \neg A(x)$$

Application Example

De Morgan and other Laws Application in Mathematical Statements

$$\neg \forall x((x>0 \Rightarrow x+y > 0) \wedge \exists y (y<0))$$

$\equiv$ (by De Morgan's Law)

$$\exists x \neg ((x>0 \Rightarrow x+y > 0) \wedge \exists y (y < 0))$$

$\equiv$ (by De Morgan's Law and 1., 2., 3., 4.)

$$\exists x((x>0 \wedge x+y \leq 0) \vee \forall y(y \geq 0))$$

We used

1. $\neg (A \Rightarrow B) \equiv (A \wedge \neg B)$, 2. $\neg(A \wedge B) \equiv (\neg A \vee \neg B)$
3. $\neg(x + y) > 0) \equiv x+y \leq 0$
4. $\neg \exists y (y <0) \equiv \forall y \neg (y < 0)$

   $\equiv \exists y (y \geq 0)$

Math Statement -to -Logic Formula

Mathematical statement

$\neg \forall x((x<0 \Rightarrow x+y > 0) \wedge \exists y\, (y<0))$

Corresponding Logic Formula is

$\neg \forall x((P(x,c) \Rightarrow R(f(x,y),c)) \wedge \exists y\, P(y,c))$

More general; $A(x)$, $B(x)$ any formulas

$\neg \forall x((A(x) \Rightarrow B(x,y)) \wedge \exists y\, A(y))$

$\equiv \exists x \neg((A(x) \Rightarrow B(x,y)) \wedge \exists y\, A(y))$

$\equiv \exists x((A(x) \wedge \neg B(x,y)) \vee \neg \exists y\, A(y))$

$\equiv \exists x\, ((A(x) \wedge \neg B(x,y)) \vee \forall y \neg A\, (y))$

Distributivity Laws

1. $\exists x(A(x) \vee B(x)) \equiv (\exists x\, A(x) \vee \exists x\, B(x))$

Existential quantifier is distributive over $\vee$

What we write as $(\exists x, \vee)$

2. $\forall x\, (A(x) \wedge B(x)) \equiv (\forall x\, A(x) \wedge \forall x\, B(x))$

Universal quantifier is distributive over $\wedge$, what we write as $(\forall x, \wedge)$

Existential quantifier is distributive over $\wedge$ **only in one direction:**

3. $\exists x(A(x) \wedge B(x)) \Rightarrow (\exists x\, A(x) \wedge \exists x\, B(x))$

Distributivity Laws

We show the inverse implication

$(\exists x\, A(x) \wedge \exists x\, B(x)) \Rightarrow \exists x(A(x) \wedge B(x))$

is NOT a predicate tautology;

It means that **it is not true**, that the implication

$(\exists x\, A(x) \wedge \exists x\, B(x)) \Rightarrow \exists x(A(x) \wedge B(x))$

holds for **any** $X \neq \phi$ and for **any** $A(x)$, $B(x)$ defined in the set X

To prove it **we have to show** that

there are $X \neq \phi$, $A(x)$, $B(x)$ defined in $X \neq \phi$ for which this implication is **FALSE**

Not a Tautology

The formula

$(\exists x\, A(x) \wedge \exists x\, B(x)) \Rightarrow \exists x(A(x) \wedge B(x))$

Is not a predicate tautology

Here is a **counter- example**

Take: X = R (real numbers),

A(x): $x > 0$ and B(x): $x<0$ we get that

$\exists x\,(x>0) \wedge \exists x(x<0)$ is a **true** statement in R

and

$\exists x(x>0 \wedge x< 0)$ is a **false** statement in R

Distributivity Laws

Universal quantifier is distributive over $\vee$ in **only one direction**:

4. $((\forall x\, A(x) \vee \forall x\, B(x)) \Rightarrow \forall x(A(x) \vee B(x)))$

Here is the other direction implication counter- example

Take: **X=R** and **A(x): x < 0, B(x): x ≥ 0**

$\forall x\,(x<0 \vee x \geq 0)$ is a **true** statement in **R** (real numbers) and

$\forall x(x<0) \vee \forall x(x \geq 0)$ is a **false** statement in **R**

Distributivity Laws

Universal quantifier is distributive over $\Rightarrow$ in **one direction only**:

5. $(\forall x(A(x) \Rightarrow B(x)) \Rightarrow (\forall x\, A(x) \Rightarrow \forall x\, B(x)))$

Other direction implication **counter example**:

Take: **$X = R$**, **$A(x): x < 0$** and **$B(x): x+1 > 0$**

$(\forall x(x < 0) \Rightarrow \forall x(x+1 > 0))$ is a **True** statement in set **R** of real numbers and

$\forall x(x < 0 \Rightarrow x+1 > 0)$ is a **False** statement:

take $x = -2$, we get $(-2 < 0 \Rightarrow -2+1 > 0)$ False

Introduction and Elimination Laws

B - Formula without free variable x

6. $\forall x(A(x) \Rightarrow B) \equiv (\exists x\, A(x) \Rightarrow B)$
7. $\exists x(A(x) \Rightarrow B) \equiv (\forall x\, A(x) \Rightarrow B)$
8. $\forall x(B \Rightarrow A(x)) \equiv (B \Rightarrow \forall x\, A(x))$
9. $\exists x(B \Rightarrow A(x)) \equiv (B \Rightarrow \exists x\, A(x))$

Introduction and Elimination Laws

B - **Formula without free variable x**

10. $\forall x(A(x) \vee B) \equiv (\forall x A(x) \vee B)$

11. $\forall x(A(x) \wedge B) \equiv (\forall x A(x) \wedge B)$

12. $\exists x(A(x) \vee B) \equiv (\exists x A(x) \vee B)$

13. $\exists x(A(x) \wedge B) \equiv (\exists x A(x) \wedge B)$

Remark: we prove **6 -9** from **10 – 13** + de Morgan + definability of implication

TRUTH SETS

We use truth sets for predicates to define an intuitive semantics for predicate logic

Given a set $X \neq \phi$ and a predicate $P(x)$, the set

$$\{x \in X: P(x)\}$$

is called a truth set for the predicate $P(x)$ in the domain $X \neq \phi$

Truth Sets, Interpretations

Example

Take $P(x)$ as $x+1 = 3$

– **it is called an interpretation of** $P(x)$ in a set **$X \neq \phi$**

Let $X=\{1, 2, 3\}$ then the **truth set** for $P(x)$ is

$$\{x \in X: P(x)\} = \{x \in X: x+1 = 3\} = \{2\}$$

and we say that $P(x)$ is **TRUE** in the set X **under the interpretation** $P(x): x+1 = 3$

TRUTH SETS semantics for Connectives

We use truth sets for predicates **always for** $X \neq \phi$

Conjunction:

$\{x \in X: (P(x) \wedge Q(x))\} = \{x: P(x)\} \wedge \{x: Q(x)\}$

Truth set for conjunction $(P(x) \wedge Q(x))$ is the set **intersection** of truth sets for its components.

Disjunction:

$\{x \in X: (P(x) \vee Q(x))\} = \{x: P(x) \vee \{x: Q(x)\}$

Truth set for disjunction $(P(x) \vee Q(x))$ is the set **union** of truth sets for its components.

Negation:

$\{x \in X: \neg P(x)\} = X - \{x \in X: P(x)\}$

$\neg$ is the negation

and $-$ is the **set complement** relative to X

Truth sets semantics for Connectives

Implication:

$\{x \in X: (P(x) \Rightarrow Q(x))\}$ = $X - \{x:P(x)\} \vee \{x : Q(x)\}$

$= \{x: \neg P(x)\} \vee \{x : Q(x)\}$

Example:

$\{x \in N: n>0 \Rightarrow n^2 < 0\} = \{x \in N\ x \leq 0\} \vee \{x \in N : n^2 < 0\}$

$= \phi \vee \phi = \phi$

Truth Sets Semantics for Universal Quantifier

Definition:

$$\forall x\, A(x) = T \quad \text{iff} \quad \{x \in X: A(x)\} = X$$

where

$X \neq \phi$ and $A(x)$ is any formula with a free variable x

Definition:

$$\forall x\, A(x) = F \text{ iff} \quad \{x \in X: A(x)\} \neq X$$

where

$X \neq \phi$ and $A(x)$ is any formula with a free variable x

Truth Sets semantics for Existential Quantifier

Definition:

$$\exists x\, A(x) = T \quad (\text{in } x \neq \phi) \quad \text{iff} \quad \{x \in X : A(x)\} \neq \phi$$

Definition:

$$\exists x\, A(x) = F \quad (\text{in } x \neq \phi) \quad \text{iff} \quad \{x \in X : A(x)\} = \phi$$

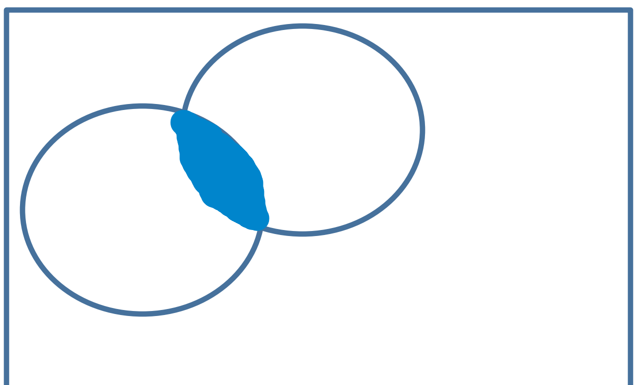
Where $X \neq \phi$ and **$A(x)$ is a formula with a free variable x**

Venn Diagrams For Existential Quantifier and Conjunction

∃x(A(x) ∧ B(x))=T iff **{x:A(X)} ∧ {x:B(x)} ≠ Φ**

Picture

X≠ Φ observe that **{x:A(X)}** ≠ Φ and **{x:B(x)} ≠ Φ**

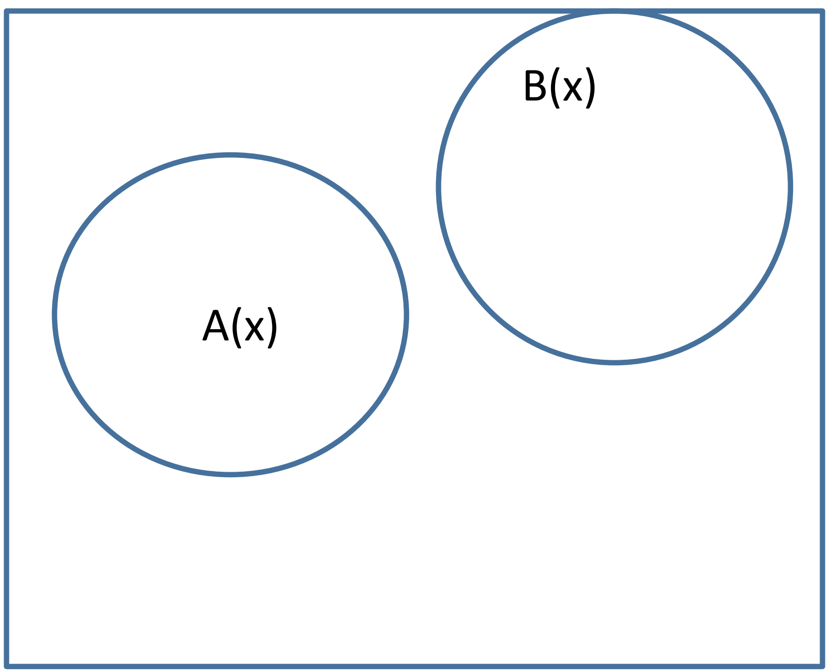


Venn Diagrams For Existential Quantifier and Conjunction

$\exists x(A(x) \wedge B(x)) = F$ iff $\{x{:}A(x)\} \wedge \{x{:}B(x)\} = \Phi$

Picture

X≠ Φ



Remember {x:A(x)}, {x:B(x)} now can be Φ!

X≠ Φ

Venn Diagrams For Universal Quantifier and Implication

Observe that

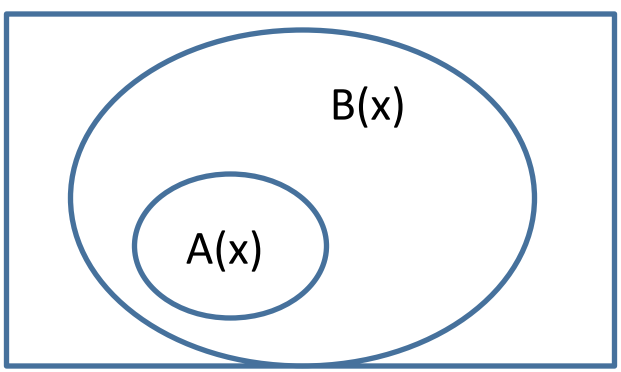
$\forall x\,(A(x) \Rightarrow B(x)) = T$ iff $\{x \in X : A(x) \Rightarrow B(x)\} = X$

Iff

$\{x:A(x)\} \subseteq \{x:B(x)\}$

$X \neq \Phi$

Picture



Remember that {x:A(x)}, {x:B(x)} now can be Φ!

Exercise

Draw a picture for a situation where (in $X \neq \Phi$)

1. $\exists x\, P(x) = T$
2. $\exists x\, Q(x) = T$
3. $\exists x(P(x) \land Q(x)) = F$
4. $\forall x\, (P(x) \lor Q(x) = F$

Exercise Solution

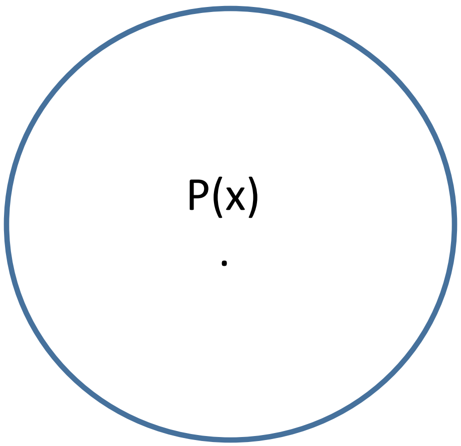
1. $\exists x\, P(x) = T$ iff $\{ x:P(x)\} \neq \Phi$

2. $\exists x\, Q(x) = T$ iff $\{x:Q\,(x)\} \neq \Phi$

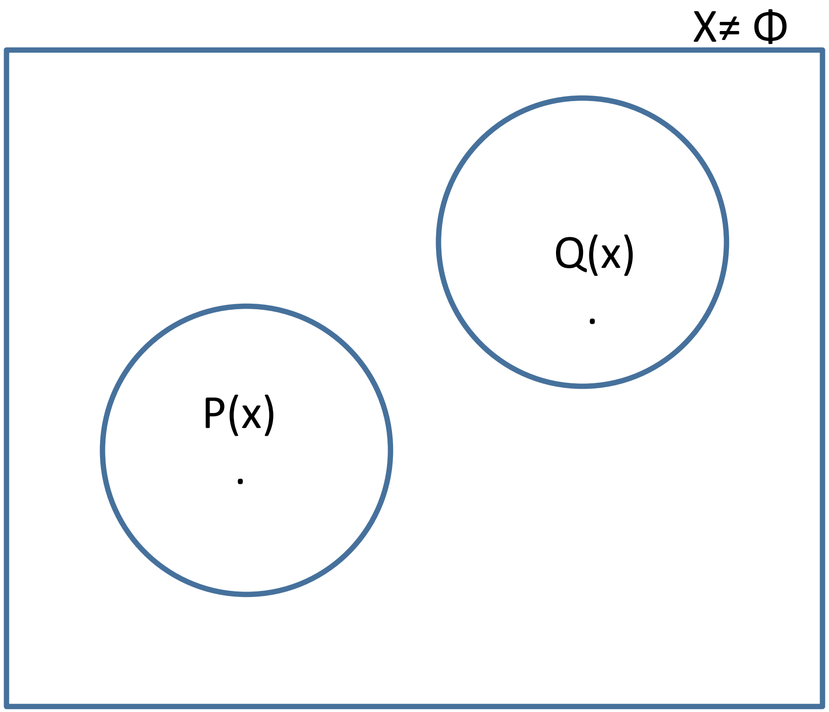
3. $\exists x(P(x) \wedge Q(x)) = F$ iff $\{x: P(x)\} \wedge \{x: Q(x)\} = \Phi$

4. $\forall x\, (P(x) \vee Q(x) = F$ iff $\{x:P(x)\} \vee \{ x:Q(x)\} \neq X$

Picture:



Denotes {x: P(x)} ≠ Φ



Proving Predicate Tautologies with **TRUTH Sets**

Prove that

$$\models (\forall x\, A(x) \Rightarrow \exists x\, A(x))$$

Proof:

Assume that not true

(Proof by contradiction) i.e. that there are $X \neq \Phi$, $A(x)$ such that.

$(\forall x\, A(x) \Rightarrow \exists x\, A(x)) = \mathbf{F}$

iff $\forall x\, A(x) = T$ and $\exists x\, A(x) = F$ $\quad\quad (A \Rightarrow B) = F$

iff $X \neq \phi$ and

$\{x \in X : A(x)\} = X$ and $\{x \in X : A(x)\} = \phi$

iff $X = \phi$

Contradiction with $X \neq \phi$, hence proved.

Proving Predicate Tautologies with TRUTH Sets

Prove:

$$\neg \forall x\, A(x) \equiv \exists x\, \neg A(x)$$

Case1: $\exists x \neg A(x) = T$ in $X \neq \phi$ **iff** $\{x: \neg A(x)\} \neq \phi$ **iff** $X - \{x: A(x)\} \neq \phi$ **iff** $\{x: A(x)\} \neq X$ **iff** $\forall x\, A(x) = F$ **iff** $\neg \forall x\, A(x) = T$

Case2: $\exists x \neg A(x) = F$ in $X \neq \phi$ **iff** $\{x: \neg A(x)\} = \phi$ **iff** $X - \{x: A(x)\} = \phi$ **iff** $\{x: A(x)\} = X$ **iff** $\forall x\, A(x) = T$ **iff** $\neg \forall x\, A(x) = F$

Prove

$\exists x(A(x) \lor B(x)) \equiv \exists x\, A(x) \lor \exists x\, B(x)$

Case 1: $\exists x(A(x) \lor B(x)\} = T$ iff

$\{x: (A(x) \lor B(x)\} \neq \phi$ (definition)

$= \{x: (A(x)\} \lor \{x: (B(x)\} \neq \phi$ iff

$\{x: A(x)\} \neq \phi$ or $\{x: B(x)\} \neq \phi$ iff

$= \exists x\, A(x)=T$ or $\exists x\, B(x)=T$

We used: for any sets, $A \lor B \neq \phi$ iff

$A \neq \phi$ or $B \neq \phi$

Case2 – similar

Russell's Paradox

We assumed in our approach that for any statement A(x)

the TRUTH set

$\{x \in X: A(x)\}$ exists

Russell Antinomy showed that that technique of TRUTH sets is **not sufficient**

This is why we need a proper semantics!